

# ILLUMINATION FROM WITHIN FOR UNBOUNDED CONVEX BODIES

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ILLUMINATION FROM WITHIN FOR UNBOUNDED CONVEX BODIES

(Presented by Academician P. S. Aleksandrov on 2 III 1970)

Let  $K$  be a convex body of the  $n$ -dimensional Euclidean space  $E^n$ ; let  $\text{bd } K$  be its boundary and  $\text{int } K$  its interior. A point  $y \in \text{bd } K$  will be called **illuminated from within** by a point  $x \in \text{bd } K$  if  $x \neq y$  and the interval  $(x, y)$  is contained in  $\text{int } K$ . A set  $N \subset \text{bd } K$  will be called **illuminated from within** by a set  $M \subset \text{bd } K$  if every point  $y \in N$  is illuminated from within by some point  $x \in M$ . We shall assume throughout that  $\text{bd } K \neq \emptyset$ .

**Theorem 1.** *The boundary  $\text{bd } K$  of a convex unbounded body  $K \subset E^n$  is illuminated from within if and only if  $K$  is not a cone.*

**Proof.** If  $K$  is a cone with vertex  $a$ , then, evidently, the point  $a$  is not illuminated from within by any point  $x \in \text{bd } K$ .

Conversely, if the boundary  $\text{bd } K$  of the body  $K$  is not illuminated from within, then there exists a point  $z \in \text{bd } K$  that is not illuminated from within by any point  $x \in \text{bd } K$ . Let  $y \in \text{int } K$ . If the ray  $zy$  did not belong entirely to the body  $K$ , then on this ray there would be found such a point  $x \in \text{bd } K$  that  $y \in (x, z)$ . But this would mean that the point  $z$  is illuminated from within by the point  $x \in \text{bd } K$ , contradicting the assumption. Thus the ray  $zy$  is contained in the body  $K$ . Consequently,  $\{z\} \cup \text{int } K$  is a cone, and therefore, passing to the closure, we find that  $K$  is a cone (with vertex at the point  $z$ ).

Theorem 1 resolves the question under what condition the entire boundary  $\text{bd } K$  of the body  $K$  is illuminated from within by some set  $M \subset \text{bd } K$ . The least cardinality of an illuminating set  $M$  will be denoted by  $p(K)$ . It is known <sup>(3, 5)</sup> that for every bounded convex body  $K \subset E^n$  the inequalities  $2 \leq p(K) \leq n + 1$  hold. Furthermore <sup>(5)</sup>, for every natural  $p$  satisfying the inequalities  $2 \leq p \leq n + 1$ , there exists such a bounded convex body  $K \subset E^n$  that  $p(K) = p$ , and the equality  $p(K) = n + 1$  holds only for the  $n$ -dimensional simplex. In this note the indicated results are extended to the case of unbounded convex bodies.

A point  $x \in \text{bd } K$  will be called **regular** if, first, through  $x$  there passes a **unique** supporting hyperplane  $\Gamma_x$  of the body  $K$ , and, second, the point  $x$  is an **interior point** of the convex set  $\Gamma_x \cap \text{bd } K$  (relative to the carrier plane of

this set). If  $x$  is a regular point, then the closed convex set  $F_x = \Gamma_x \cap \text{bd } K$  will be called a **regular face**.

**Lemma 1.** *If  $x$  is a regular point and  $y$  is an interior point of the corresponding regular face  $F_x$  (relative to its carrier plane), then  $y$  is also a regular point and  $F_y = F_x$ .*

**Lemma 2.** *The set  $M$  of all regular points is dense in the boundary of the body  $K$ , i.e.  $\overline{M} = \text{bd } K$ .*

For the proof of these lemmas see <sup>(2)</sup>, p. 110, and <sup>(1)</sup>, p. 24.

**Lemma 3.** *An unbounded convex body  $K \subset E^n$  is a cone if and only if, for any  $n$  regular points  $x_1, x_2, \dots, x_n$ , the corresponding regular faces  $F_{x_1}, F_{x_2}, \dots, F_{x_n}$  have a nonempty intersection.*

In the proof only the sufficiency of this condition is needed.

First of all it is established that if the intersection of any  $n$  regular faces is nonempty, then the intersection of all regular faces is nonempty (for this one considers all possible intersections of regular faces with one of the planes  $\Gamma_x$ , and applies Helly's theorem <sup>(3)</sup> to these intersections). Next it is shown (using Lemma 2) that if  $a$  is a common point of all regular faces, then  $\text{bd } K$  is a cone with vertex  $a$ . From this it follows without difficulty that  $K$  is a convex cone with vertex  $a$ .

**Theorem 2.** *For every unbounded convex body  $K \subset E^n$  distinct from a cone, the inequalities  $2 \leq p(K) \leq n$  hold.*

**Proof.** Since the body  $K$  is distinct from a cone, by Lemma 3 there exist  $n$  regular points  $x_1, x_2, \dots, x_n \in \text{bd } K$  such that the corresponding regular faces  $F_{x_1}, F_{x_2}, \dots, F_{x_n}$  have empty intersection. If a point  $x \in \text{bd } K$  is not illuminated from within by some point  $x_i$  ( $i = 1, 2, \dots, n$ ), then the whole segment  $[x_i, x]$  lies on the boundary of the body  $K$  and, hence, in the hyperplane  $\Gamma_{x_i}$ . In particular,  $x \in \Gamma_{x_i}$ . Since, moreover,  $x \in \text{bd } K$ , it follows that  $x \in \Gamma_{x_i} \cap \text{bd } K = F_{x_i}$ . Thus, if a point  $x \in \text{bd } K$  is not illuminated from within by the point  $x_i$ , then  $x \in F_{x_i}$ , and consequently  $x$  belongs to the intersection of the faces  $F_{x_1}, F_{x_2}, \dots, F_{x_n}$ . But since the sets  $F_{x_1}, F_{x_2}, \dots, F_{x_n}$  have empty intersection, it follows that every point  $x \in \text{bd } K$  is illuminated by at least one of the points  $x_1, x_2, \dots, x_n$ . Consequently,  $p(K) \leq n$ .

The inequality  $2 \leq p(K)$  is obvious, since a single point  $x \in \text{bd } K$  cannot illuminate from within the whole boundary  $\text{bd } K$  of the body  $K$  (the point  $x$  itself will remain unilluminated from within). This completes the proof of Theorem 2.

**Theorem 3.** *For every integer  $p$  satisfying the inequalities  $2 \leq p \leq n$ , there exists an unbounded convex body  $K \subset E^n$  such that  $p(K) = p$ .*

**Proof** of Theorem 3 is carried out by the same methods as for bounded convex bodies (see <sup>(5)</sup>), but cylinders are considered instead of balls. Namely, if  $L$  is

a convex set with supporting hyperplane  $\Gamma \subset E^n$  and if  $a \notin \Gamma$ , then, as is not hard to prove, for the cone  $aL$  with base  $L$  and vertex  $a$  the relation

$$p(aL) = p(L) + 1.$$

Therefore, if  $C$  is such a convex body of dimension  $n - p + 2$  for which  $p(C) = 2$ , then, forming  $p - 2$  times a cone over this body, we arrive at an  $n$ -dimensional body  $K$  satisfying the relation  $p(K) = p$  (here  $p$  may take the values  $2, 3, \dots, n$ ). In note <sup>(5)</sup>, an  $(n - p + 2)$ -dimensional ball was used as  $C$ . If, however, one takes as  $C$  a cylinder, i.e., the  $r$ -neighborhood of a straight line in  $(n - p + 2)$ -dimensional space, then this construction leads us to the required unbounded body. Thus Theorem 3 is proved.

Finally, let us note that an unbounded body  $K \subset E^n$  for which  $p(K)$  assumes its maximal possible value is no longer unique, as it was in the case of a bounded body. Indeed, if instead of  $C$  in the proof of Theorem 3 one considers some two-dimensional unbounded convex polygon having two unbounded parallel sides, then the body  $K$  to which we arrive, as is not hard to trace from the proof of Theorem 3, will satisfy the relation  $p(K) = n$ . But there are infinitely many such polygons  $C$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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