

THE RELATION BETWEEN TOPOLOGICAL ENTROPY AND METRIC ENTROPY

E. I. Dinaburg

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.22422>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.21

MATHEMATICS

E. I. DINABURG

THE RELATION BETWEEN TOPOLOGICAL ENTROPY AND METRIC ENTROPY

(Presented by Academician A. N. Kolmogorov, 29 V 1969)

This article studies the connection between the notions of topological entropy, ε -entropy, and metric entropy. Some examples are considered.

I. The connection between topological entropy and ε -entropy. Let $M = (E, \rho)$ be a compact metric space (E is the set of points, $\rho(x, y)$ is the distance), and let $T : M \rightarrow M$ be an arbitrary homeomorphism. Denote by $h(T | M)$ the topological entropy of the transformation T on the space M ⁽⁴⁾. Introduce on E new metrics

$$\rho_n(x, y) = \max_{0 \leq i < n} \rho(T^i x, T^i y).$$

Let $H_n(\varepsilon)$ be the ε -entropy of the space $M_n = (E, \rho_n)$ in the sense of ⁽¹⁾. A. N. Kolmogorov (unpublished) suggested the idea of relating the topological entropy of a homeomorphism with the asymptotics of the function $H_n(\varepsilon)$.

Theorem 1. *The equality holds*

$$h(T | M) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{H_n(\varepsilon)}{n}.$$

The proof of this theorem is not difficult, and therefore we do not give it. Theorem 1 often proves useful in estimating from below the topological entropy of a particular dynamical system (see below).

II. The connection between topological entropy and metric entropy. All measures considered below are assumed to be normalized and defined on the σ -algebra of all Borel subsets of some compact metric space.

We shall denote by $h_\mu(T | M)$ the metric (with respect to the measure μ) entropy of the transformation T on M , and by $h_\mu(T | M; \xi)$ the metric (with respect to the measure μ) entropy per unit time of a partition ξ of the space M ⁽²⁾.

In ⁽⁴⁾ the hypothesis was put forward that

$$h(T \mid M) = \sup_{\mu} h_{\mu}(T \mid M),$$

where the supremum is taken over all Borel measures μ invariant with respect to T . Below the validity of this hypothesis is proved for homeomorphisms of an arbitrary compact space of finite dimension.

Theorem 2. *Let M be an arbitrary compact metric space of finite dimension m , and let $T : M \rightarrow M$ be its homeomorphism. Then*

$$h(T \mid M) = \sup_{\mu} h_{\mu}(T \mid M),$$

where the supremum is taken over all Borel measures μ invariant with respect to T .

Proof. First we shall prove that $h(T \mid M) \leq \sup_{\mu} h_{\mu}(T \mid M)$. Suppose that $h(T \mid M) < \infty$. (The case $h(T \mid M) = \infty$ is treated similarly.) It is enough to prove that for every positive integer r there exists a Borel measure μ_r , invariant with respect to T^r , such that

$$h_{\mu_r}(T^r \mid M) \geq h(T^r \mid M) - b_r = rh(T \mid M) - b_r,$$

where $|b_r| < b$. Indeed, the measure

$$\mu'_r = \frac{1}{r} \sum_{i=0}^{r-1} T^i \mu_r$$

is invariant with respect to T , and

$$h_{\mu'_r}(T \mid M) = \frac{1}{r} h_{\mu'_r}(T^r \mid M) \geq \frac{1}{r} h_{\mu_r}(T^r \mid M) \geq h(T \mid M) - \frac{b_r}{r}.$$

The desired inequality follows from the fact that $b_r/r \rightarrow 0$ as $r \rightarrow \infty$.

For any finite covering $\{V_i\}$ of the space M by closed sets $V_1, \dots, V_N$, consider the set A of all sequences $\{i_k\}$ ($-\infty < k < \infty$), composed of the symbols $1, \dots, N$, such that

$$\bigcap_{-\infty}^{\infty} T^{kr} V_{i_k}$$

is nonempty.

Lemma 1. The set A is closed in the space Ω^N of all sequences composed of the symbols $1, \dots, N$, and is invariant with respect to the shift S by one symbol to the right.

Lemma 2. There exists $\delta(T, r)$ such that, if $\text{diam}\{V_i\} < \delta(T, r)$,

$$h(S \mid A) > h(T^r \mid M) - 1.$$

Lemma 3. On A there exists a Borel measure μ , invariant with respect to S , such that

$$h_\mu(S \mid A) = h(S \mid A).$$

(See also ⁽⁸⁾.)

The central point of the proof is the following assertion.

Lemma 4. There exist a covering $\{V_i\}$ of the space M by closed sets $V_1, \dots, V_N$, a compact Ω , and a homeomorphism $R : \Omega \rightarrow \Omega$ such that:

- 1) each element of the covering $\{V_i\}$ intersects no more than 3^{2m+1} elements of the same covering;
- 2) the space of sequences A , constructed by the method described above from the covering $\{V_i\}$, satisfies the conditions of Lemmas 1-3;
- 3) there exists a continuous mapping $p : \Omega \rightarrow A$ such that $p(\Omega) = A$ and $pR = Sp$;
- 4) there exists a continuous mapping $p_0 : \Omega \rightarrow M$ such that $p_0(\Omega) = M$ and $p_0R = T^r p_0$;
- 5) for any sequence $\omega = \{i_k\} \in A$,

$$p_0 p^{-1}(\omega) \subset \bigcap_{-\infty}^{\infty} T^{kr} V_{i_k}.$$

Lemma 5. On Ω there exists a Borel measure ν , invariant with respect to R , such that for every Borel set $B \subset A$,

$$\mu(B) = \nu(p^{-1}(B)).$$

Define a Borel measure μ_r on M by setting $\mu_r(B) = \nu(p_0^{-1}(B))$ for every Borel set $B \subset M$. The measure μ_r has the required properties. Indeed, from Lemma 4 it follows that the measure μ_r is invariant with respect to T^r . Consider the partition α of the space A into the sets $C_1, \dots, C_N$, where $C_i = \{\omega = \{j_k\} : j_0 = i\}$. Clearly,

$$h(S \mid A) = h_\mu(S \mid A; \alpha) = h_\nu(R \mid \Omega; \xi),$$

where $\xi = p^{-1}(\alpha)$.

Let β be the partition of the space M into the sets $\widetilde{C}_1, \dots, \widetilde{C}_N$, where

$$\widetilde{C}_1 = V_1, \quad \widetilde{C}_i = V_i - \bigcup_{k=1}^{i-1} \widetilde{C}_k \quad \text{for } 1 < i \leq N.$$

Put $\eta = p_0^{-1}(\beta)$. Clearly,

$$h_{\mu_r}(T^r \mid M; \beta) = h_\nu(R \mid \Omega; \eta).$$

Further:

$$\begin{aligned} h_\nu(R \mid \Omega; \eta) &= \lim_{n \rightarrow \infty} \frac{1}{n} H(\eta^n) = \lim_{n \rightarrow \infty} \left[\frac{1}{n} H(\xi^n \eta^n) - \frac{1}{n} H(\xi^n / \eta^n) \right] \geq \\ &\geq h(S \mid A) - \lim_{n \rightarrow \infty} \frac{1}{n} H(\xi^n / \eta^n) \geq h(S \mid A) - (2m+1) \log 3. \end{aligned}$$

Consequently,

$$h_{\mu_r}(T^r \mid M) > h(T^r \mid M) - b_r, \quad \text{where } |b_r| < (2m+1) \log 3 + 1.$$

Lemma 6. *For any metric compactum M of dimension m ($m < \infty$), any homeomorphism $T : M \rightarrow M$, and any Borel measure μ invariant with respect to T , the inequality $h(T \mid M) \geq h_\mu(T \mid M)$ holds.*

The proof of Lemma 6 follows from the fact that in any open cover of the compactum M one can inscribe a cover each element of which intersects no more than 3^{2m+1} elements of the given cover.

Theorem 2 follows from Lemmas 1-6.

G. A. Margulis (unpublished) independently proved Lemma 6 for compacta of a more general nature.

Not for every homeomorphism T of a compactum M does there exist an invariant measure μ such that $h(T \mid M) = h_\mu(T \mid M)$. In [9] an example is constructed of a homeomorphism of a zero-dimensional compactum that does not have this property.

Below it is assumed that M is a metric space of finite dimension.

Corollary 1. *The entropy $h(T | M)$ is equal to the topological entropy of the restriction of the transformation T to the set of nonwandering points.*

This assertion in a more general situation was proved by Bowen [5].

Corollary 2. *For a flow $\{S_t\}$ on M , for any t*

$$h(S_t | M) = |t|h(S_1 | M).$$

Corollary 2 was previously proved by A. M. Stepin (unpublished).

Corollary 3. *Let $M = X \times Y$, and let $T : M \rightarrow M$ be a homeomorphism identical on the base X , i.e. $T(x, y) = (x, T_x y)$. Then*

$$h(T | M) = \sup_x h(T_x | Y).$$

This assertion is a simple consequence of the fact that the equality $h(T | M) = \sup h_\mu(T | M)$ also holds in the case when the supremum is taken only over all invariant ergodic Borel measures μ .

Example A. The topological entropy of the geodesic flow on a compact surface of constant negative curvature coincides with its metric entropy with respect to the usual Riemannian measure.

The topological entropy of the horocycle flow on such a surface is equal to zero.

Example B. Let M be an arbitrary compact orientable surface of genus $p > 1$, let $\widetilde{M}$ be the space of line elements to M , and let S_t be the geodesic flow on $\widetilde{M}$.

Theorem 3. *The topological entropy $h(S_t | \widetilde{M})$ is positive, and there exists a measure μ , invariant with respect to S_t , for which $h_\mu(S_t | \widetilde{M}) > 0$.*

The proof is based on the study of geodesics of class A and on results of M. Morse concerning such geodesics [6, 7].

Example C. Motion of a particle on a torus in the field of a repelling potential. Let the motion of a point be defined by Hamilton's equations with Hamiltonian function $H = \dot{x}^2/2 + \dot{y}^2/2 + U(r)$, where $U(r)$ is a smooth function for $r > 0$, with $U(r) \equiv 0$ for $r \geq r_0$, $\partial U/\partial r \leq 0$, and $U(r) \rightarrow \infty$ as $r \rightarrow 0$ (here $r^2 = x^2 + y^2$, and the coordinates x, y are considered modulo 1).

Theorem 4. For every fixed $\bar{H} \leq H_0$ and $r_0 \leq r(H_0)$, the flow defined by the motion of a point on the surface of constant energy $H = \bar{H}$ has positive topological entropy.

In the work of Ya. G. Sinai (3), conditions were found under which, for systems of this type, the Lebesgue measure has positive entropy.

In conclusion, the author expresses gratitude to A. N. Kolmogorov for valuable comments, and to Ya. G. Sinai and B. M. Gurevich for useful discussions of a certain finite set.

Institute of Chemical Physics
Academy of Sciences of the USSR
Moscow

Received
20 V 1969

REFERENCES

- ¹ A. N. Kolmogorov, V. M. Tikhomirov, UMN, 14, no. 2, 3 (1959).
- ² V. A. Rokhlin, UMN, 22, no. 5, 3 (1967).
- ³ Ya. G. Sinai, DAN, 153, no. 6, 1261 (1963).
- ⁴ R. L. Adler, A. G. Konheim, M. H. McAndrew, Trans. Am. Math. Soc., 114, no. 2, 309 (1965).
- ⁵ R. Bowen, Preprint, Warwick, Coventry, 1969.
- ⁶ M. Morse, Trans. Am. Math. Soc., 26, no. 1, 25 (1924).
- ⁷ M. Morse, J. Math. Pures Appl., 14, fasc. 1, 49 (1935).
- ⁸ W. Parry, Trans. Am. Math. Soc., 122, no. 1, 368 (1968).
- ⁹ B. M. Gurevich, DAN, 187, no. 2 (1969).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.