

On Multiple Regular Bases in Köthe Space

M. M. Dragilev

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.22313>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1970. Volume 193, No. 4

UDC 513.881

MATHEMATICS

M. M. DRAGILEV

ON MULTIPLE REGULAR BASES IN KÖTHE SPACE

(Presented by Academician L. V. Kantorovich on 26 IX 1969)

1. Let E be a Köthe space*; let (x_n) be an arbitrary absolute basis of it; let (y_n) be one of those absolute bases in E which can be obtained as a result of the following three operations: a) the mapping $(x_n) \rightarrow (Tx_n)$, where T is an isomorphism of the space E onto itself; b) multiplication of the elements Tx_n by numbers λ_n , $\lambda_n \neq 0$ ($n = 1, \dots$); c) permutation of the terms of the sequence $(\lambda_n Tx_n)$. The bases (x_n) and (y_n) are called, respectively, equivalent, pre-equivalent, and quasi-equivalent. Up to the present time the question has not been completely solved (see, for example, ⁽¹⁾): in what case are all absolute bases of a space in one of the listed equivalence relations?

It is not difficult to see that any two bases in E are equivalent if and only if the space is finite-dimensional (here it is essential that the topology in E can be given by a countable set of norms ⁽³⁾). Analogously, all absolute bases in E are pre-equivalent if and only if E is normable (i.e., degenerates into the Banach space l_1)**. It is not known, however, whether all absolute bases of a nondegenerate space E are quasi-equivalent. A positive answer has been obtained only for some special classes of nuclear and Montel Köthe spaces possessing a regular*** basis ⁽³⁻⁹⁾.

In the present article a more general case is considered. Let (x_n) be an arbitrary sequence in E ; let μ be the minimal number of its regular subsequences such that each element x_n is contained in one and only one of them. Put $m = \inf \mu$, where the infimum is taken over the set of all possible permutations of the elements x_n ($n = 1, 2, \dots$). A sequence for which $m \neq 1$ cannot be made regular by a permutation of its elements. It is natural to call it an m -fold regular sequence if m is finite and $\mu = m$. We shall extend this definition also to the case when $m = 1$ or $m = \infty$. As will be shown, the multiplicity m of an arbitrary basis in a nuclear space E depends only on the space. In other words, each nuclear Köthe space belongs to one and only one of the classes: $\mathcal{N}_s = \{E : m = s\}$,

$1 \leq s \leq \infty$. All the classes $\mathcal{N}_s$ are nonempty. Any of them, with the exception, perhaps, of $\mathcal{N}_\infty$, contains spaces all of whose bases are quasi-equivalent.

2. Let (x_n) be a sequence in E , the elements of which are not equal to zero; let $(t_n) = t$ be an arbitrary numerical sequence and

$$|t|_p = \sum_n |t_n| \|x_n\|_p \quad (p = 1, 2, \dots).$$

We agree to denote by $[x_n]$ the Köthe space

$$\{t : |t|_p < \infty, p = 1, \dots\}$$

(with the topology given by the system—

* A Köthe space is a complete countably normed space with an absolute Schauder basis.

** A more particular assertion is proved in ⁽¹⁰⁾.

*** A sequence (x_n) , $x_n \in E$ ($1 \leq n < \nu \leq \infty$), is called regular if there exists a defining system of norms in E such that, for any $p, q = 1, 2, \dots$, the sequence of numbers $\|x_n\|_p / \|x_n\|_q$ ($1 \leq n < \nu \leq \infty$) is monotone ⁽⁷⁾.

by the system of norms $|\cdot|_p$. If E is a Montel space with an absolute regular basis (x_n) , then denote

$$K(E) = \{[\lambda_n x_{k_n}] : k_n \rightarrow \infty, \lambda_n > 0, n = 1, \dots\}.$$

Suppose that all regular bases in E are pre-equivalent (in this case $K(E)$ does not depend on the choice of basis) and, moreover, that every space contained in $K(E)$ has the analogous property. The class $\mathcal{E}$ of all spaces E under consideration is nonempty (it contains, in particular, spaces of type (d_i) , $i = 1, 2$ ⁽⁷⁾). Moreover, $E \in \mathcal{E}$ if and only if $K(E) \subset \mathcal{E}$.

For arbitrary spaces $E_i \in \mathcal{E}$ ($i = 1, 2$), put

$$K(E_1) \oplus K(E_2) = \{G_1 \oplus G_2 : G_i \in K(E_i), i = 1, 2\}.$$

Let R be a subset of the class $\mathcal{E}$, closed with respect to the operation $\oplus$. We say that the function $K(E)$ is additive on R , if $K(E_1 \oplus E_2) = K(E_1) \oplus K(E_2)$ for any $E_i \in R$ ($i = 1, 2$).

Definition 1. A maximal subset $R \subset \mathcal{E}$, closed with respect to the operation $\oplus$, will be called a **Riss class** if the function $K(E)$ is additive on R .

For example, the set R_0 (respectively, R_∞) of all Köthe spaces that are finite (infinite) centers of Riss scales ⁽⁵⁾ is a Riss class. Let, in general, E be an arbitrary space of type (d_1) or (d_2) . The following holds.

Theorem 1. *The set $K(E)$ is a Riss class if and only if E is isomorphic to each of its subspaces of finite codimension.*

In what follows we call a Riss class **complete** if there exists a space $E \in R$ such that $K(E) = R$.

Let us note some corollaries. Let $f(u)$ be a nondecreasing odd function of a real argument, logarithmically convex for $u \geq 0$. In ⁽⁷⁾ a broad class of Montel Köthe spaces $L_f(b, r) = [(\delta_{nj})_{j=1}^\infty]$ is considered, for which

$$|(\delta_{nj})_{j=1}^\infty|_p = \exp f(r_p b_n) \quad (r_p \uparrow r),$$

where $-\infty < r \leq \infty$, $b = (b_n)$, and $b_n \uparrow \infty$. By definition,

$$(f)_\sigma = \{L_f(b, r) : b_n \uparrow \infty, r = \sigma\} \quad (\sigma = -1, 0, 1, \infty).$$

There, countable families of pairwise disjoint classes $(f)_\sigma$ are also singled out.

Corollary 1. *The set $(f)_\sigma$ is a complete Riss class.*

Corollary 2. *The cardinality of a maximal set of pairwise disjoint complete Riss classes is at least c .*

Corollary 3. *The set $\mathfrak{R}$ of all Riss classes is uncountable.*

We formulate necessary and sufficient conditions for the classes $R_i \in \mathfrak{R}$ ($i = 1, 2$) not to intersect. Let $T : E_1 \rightarrow E_2$ be a continuous linear operator that carries an absolute basis (x_n) of the space $E_1 \in R_1$ into an absolute basis of the space $E_2 \in R_2$.

Theorem 2. *The following assertions are equivalent:*

1°. $R_1 \cap R_2 = \emptyset$.

2°. *Whatever the spaces $E_i \in R_i$ ($i = 1, 2$), the absolute basis (x_n) in E_1 , and the operator T , there exists a subsequence (x_{j_n}) , $n = 1, 2, \dots$ such that the restriction of T to the corresponding subspace*

$$\text{span}(x_{j_n}) \subset E_1$$

*is compact.**

If the classes R_1 and R_2 are such that all continuous linear mappings of each space $E_1 \in R_1$ into any space $E_2 \in R_2$ are compact, then we shall write $R_1 > R_2$.

Definition 2. Classes $R_i \in \mathfrak{R}$ ($i = 1, 2$) will be called **essentially distinct** if $R_1 > R_2$ or $R_2 > R_1$.

* For the definition of a compact operator, see, for example, ⁽¹¹⁾.

For example, the classes R_0 and R_∞ are essentially different, and moreover $R_0 > R_\infty$ ⁽²⁾. As V. P. Zaharyuta showed (ibid.), there exist countable families of Riss classes $(f)_\sigma$, linearly ordered by the relation $(\geq)$. It is not known whether any nonintersecting classes $R_i \in \mathfrak{R}$ ($i = 1, 2$) are essentially different.

Remark. Nonintersecting subsets $K(E_i) \subset R_i$ ($i = 1, 2$) cannot be essentially different in the sense of Definition 2.

Let, further, $E = \bigoplus E_i$, where $E_i \in R_i \in \mathfrak{R}$ ($i = 1, \dots, k$), and $R_i = K(E_i)$ for $i \leq s \leq k$, i.e. all the Riss classes $R_1, \dots, R_s$ are complete, and the corresponding spaces E_i satisfy the condition of Theorem 1. Important for what follows is

Theorem 3. *If $1 \leq s \leq k$ and all R_i are distinct (respectively, $1 \leq s < k$ and $R_i \cap R_j = \emptyset$ for $i \leq s < j$), then in the space E there exists an m -fold regular basis, for which $m \geq s$ (respectively $m \geq s + 1$). Moreover, equality is attained in every case.*

3. Let $E \subset \mathcal{N}$, where $\mathcal{N}$ is the class of all nuclear Köthe spaces. As is known ⁽¹²⁾, all bases in E are absolute.

Lemma (cf. ^(7,8)). *Whatever the bases (x_n) and (y_n) in E , there exist two sequences (k_n) , $k \rightarrow \infty$, and (λ_n) , $\lambda_n > 0$ ($n = 1, 2, \dots$), such that $[y_n] = [\lambda_n x_{k_n}]$.*

We give a number of corollaries, some of which are of independent interest.

Corollary 1. *In a nuclear space with a regular basis, the multiplicity m of an arbitrary basis is equal to one.*

Corollary 2. *In a nuclear space $E \in \mathcal{E}$, all bases are quasiequivalent.*

Extend to the whole class $\mathcal{N}$ the function $K(E)$ defined on the set $\mathcal{N} \cap \mathcal{E}$ (here, as (x_n) , an arbitrary basis of the space is taken).

Corollary 3. *The function $K(E)$ does not depend on the choice of a basis in E and is a topological invariant on the class $\mathcal{N}$.*

Corollary 4. *The multiplicity m of an arbitrary basis in a nuclear space E is a constant depending only on E .*

Corollary 5. *Every space $E \in \mathcal{N}$ belongs to one and only one of the classes $\mathcal{N}_s$ ($1 \leq s \leq \infty$).*

Hence, and also from Theorems 1 (Corollary 2) and 3, it follows that

Corollary 6. *None of the classes $\mathcal{N}_s$ ($1 \leq s \leq \infty$) is empty.*

Let, further, E^j denote the subspace of the space E of codimension j for $j \geq 0$, and the direct sum of the form $E \oplus L$, where $\dim L = -j$, for $j < 0$. From Corollary 3 there follows a theorem of conditional type.

Theorem 4. Let $E = \bigoplus E_i$, where $E_i \in R_i$ ($i = 1, \dots, s$) are nuclear spaces, and $R_i \cap R_k = \emptyset$ if $i \neq k$. The following assertions are equivalent:

1°. In the space E all bases are quasiequivalent.

2°. Whatever the nuclear spaces $G_i \in R_i$ ($i = 1, 2, \dots, s$), the isomorphism $E \sim \bigoplus G_i$ holds if and only if there are j_i , $\sum_i j_i = 0$, such that $E_i \sim G_i^{j_i}$ ($i = 1, \dots, s$).

It remains for us to apply the following result.

Theorem 5 (see (2)). Let X_i and Y_i ($i = 1, 2$) be linear topological spaces such that every continuous linear mapping from X_1 to Y_2 and from Y_1 to X_2 is compact. The products $X_1 \times X_2$ and $Y_1 \times Y_2$ are isomorphic if and only if, for some finite j , $X_1 \sim Y_1^j$ and $X_2 \sim Y_2^{-j}$.

An immediate consequence of Theorems 3, 4, and 5 is

Theorem 6*. In the space $E = \bigoplus E_i$, where $E_i \in R_i$ ($i = 1, 2, \dots, s$)—

* This theorem was proved by me jointly with V. P. Zaharyuta.

nuclear Köthe spaces belonging to essentially different (ordered) Riss classes, all bases are quasi-equivalent. Moreover $E \in \mathcal{N}_s$, if $R_i = K(E_i)$ ($i = 1, \dots, s$).

Rostov State

University

Received

22 IX 1969

REFERENCES

1. M. M. Dragilev, V. P. Zakharuta, M. G. Haplanov, *On some problems of bases of analytic functions*, Rostov, 1967.
2. V. P. Zakharuta, *Functional Analysis*, 4, no. 2 (1970).
3. M. M. Dragilev, Abstracts of Reports, IV All-Union Conference on the Theory of Functions of a Complex Variable, 1958, Moscow, 1961.
4. M. M. Dragilev, *UMN*, 15, no. 2 (92) (1960).
5. B. S. Mityagin, *DAN*, 137, no. 3 (1961).
6. B. S. Mityagin, *UMN*, 16, no. 4 (100) (1961).
7. M. M. Dragilev, *Matem. sborn.*, 68, no. 2 (1965).
8. C. Bessaga, *Studia Mathematic.*, 31 (1968).

9. V. P. Zakharuta, *DAN*, 180, no. 4 (1968).
10. Yu. F. Korobeinik, Collection of student papers, Rostov State University, 1957.
11. A. Robertson, B. Robertson, *Topological Vector Spaces*, Moscow, 1967.
12. A. Dynin, B. Mitiagin, *Bull. Acad. Pol. Sci.*, 8, no. 8 (1960).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.