

HOMOGENEOUS GENERALIZED FUNCTIONS AND THE RADON TRANSFORM IN THE SPACE OF RECTANGULAR MATRICES OVER A NON-DISCRETE LOCALLY COMPACT DISCONNECTED FIELD

V. G. Chernov

1970

SovietRxiv

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.948.5

MATHEMATICS

V. G. CHERNOV

HOMOGENEOUS GENERALIZED FUNCTIONS AND THE RADON TRANSFORM IN THE SPACE OF RECTANGULAR MATRICES OVER A NON-DISCRETE LOCALLY COMPACT DISCONNECTED FIELD

(Presented by Academician I. G. Petrovskii, 15 VIII 1969)

1. Let $M_{n,m} = M_{n,m}(K)$ be the space of rectangular $m \times n$ -matrices $x = x_{(ij)}$, $n \geq m$, with entries from a non-discrete locally compact disconnected field K . Denote by $S_{n,m}$ the space of finite, locally constant functions f on $M_{n,m}$. In other words, $S_{n,m}$ is the set of functions such that: 1) $f(x) = 0$ when $|x|$ is sufficiently large, $|x| = \max_j (|x_{ij}|)$; 2) for every $x \in M_{n,m}$ there exists an integer $k(x)$ such that $f(x + x_0) = f(x)$, $|x_0| \leq q^{-k(x)*}$. Here q is the number of elements of the residue field $\mathfrak{K} = \mathcal{O}/\mathfrak{P}$, where $\mathcal{O}$ is the subring of integral elements x of the field K , i.e., elements for which $|x| \leq 1$; $\mathfrak{P}$ is the maximal ideal in $\mathcal{O}$. A natural topology is introduced in $S_{n,m}$ ⁽¹⁾, with respect to which $S_{n,m}$ is a complete linear space. By generalized functions on $M_{n,m}$ we shall henceforth mean linear continuous functionals on $S_{n,m}$.

Denote by $D_{n,m}(x)$ the maximum of the norms of the minors of order m of the matrix x^{**} . For every complex number λ , $\operatorname{Re} \lambda > 0$, define the generalized function $D_{n,m}^\lambda(x)$ by the convergent integral

$$(D_{n,m}^\lambda(x), f(x)) = \int_{M_{n,m}} D_{n,m}^\lambda(x) f(x) dx, \quad f \in S_{n,m}.$$

For $\operatorname{Re} \lambda < 0$ define $D_{n,m}^\lambda(x)$ by analytic continuation. For any λ , the function $D_{n,m}^\lambda(x)$ is a spherically symmetric homogeneous generalized function of degree of homogeneity λ in the sense of the definition given below (see item 2).

The paper studies the function $D_{n,m}^\lambda$ as an analytic function of λ . The singular points of this function are found (Theorem 1), the residues at these singular points are computed (Theorem 2), the uniqueness of a spherically symmetric

homogeneous generalized function is proved (Theorem 3), and the Fourier transform of the function $D_{n,m}^\lambda(x)$ is computed. The results are applied to the solution of a problem of integral geometry in the space $M_{n,m}$ ($n > m$) (Theorem 5). (The analogous problem of integral geometry for the spaces $M_{n,m}(\mathbf{R})$ and $M_{n,m}(\mathbf{C})$ was solved in (2).) The resulting formula contains, as a special case for $m = 1$, the inversion formula for the Radon transform in affine space over the field K .

In what follows, the field K is assumed to have characteristic different from 2 and not to contain the field of dyadic numbers $\mathbf{Q}_2$.

* In view of condition 1), $k(x)$ can always be chosen independently of x .

** In particular, for $n = m$, $D_{n,m}(x) = D_n(x) = |\det x|$.

2. In this section we shall study the generalized function $D_{n,m}^\lambda$.

Theorem 1. *The generalized function $D_{n,m}^\lambda$, considered as an analytic function of λ , is regular for all λ except the points $\lambda = -n + i$, $i = 0, 1, \dots, m-1$, at which it has simple poles.*

We give formulas for the residues of $D_{n,m}^\lambda$ at the exceptional points. Let us first introduce the necessary notation. Let

$$M_{n,m}^r = \{x \in M_{n,m} \mid \text{rang } x \leq r\}.$$

We introduce a system of coordinates in $M_{n,m}^r$. Denote by y the matrix consisting of the first r columns of the matrix $x \in M_{n,m}^r$, and by y' the matrix consisting of the remaining $m-r$ columns of the matrix x . Then in general position $\text{rang } x = r$ and $y' = y\mu$, where μ is an $r \times (m-r)$ -matrix. We shall take the elements of the matrices y and μ as coordinates on $M_{n,m}^r$; functions on $M_{n,m}^r$ will be written in the form $f(x) = f(y, y\mu)$.

Theorem 2. *The residue of the generalized function $D_{n,m}^\lambda$ at the exceptional point $\lambda = -n + r$, $r = 0, 1, \dots, m-1$, is a generalized function concentrated on the manifold $M_{n,m}^r$ and given by the formulas:*

$$\left(\text{Res}_{\lambda=-n} D_{n,m}^\lambda(x), f(x) \right) = \frac{\Phi(-n, m)}{\Phi(1, m-1) \ln q} f(0);$$

for $i = 1, 2, \dots, m-1$

$$\begin{aligned} & \left(\text{Res}_{\lambda=-n+i} D_{n,m}^\lambda(x), f(x) \right) \\ &= \frac{\Phi(-n, m) \Phi(-m, i) (\ln q)^{-1}}{\Phi(-i, i) \Phi(1, m-i-1) \Phi(-n, i)} \int f(y, y\mu) D_{n,i}^{-n+m}(y) dy d\mu *, \end{aligned}$$

where

$$dy = \prod_{i,j=1}^{n,r} dy_{ij}, \quad d\mu = \prod_{s,t=1}^{r,i,m-r} d\mu_{st},$$

$$\Phi(\lambda, k) = (1 - q^\lambda)(1 - q^{\lambda+1}) \cdots (1 - q^{\lambda+k-1}), \quad \Phi(\lambda, 0) = 1;$$

$dy_{ij}, d\mu_{st}$ are measures invariant with respect to addition on K .

We shall call a generalized function φ on $M_{n,m}$ homogeneous of degree λ if it satisfies the condition

$$(\varphi(x), f(xa^{-1})) = |\det a|^{\lambda+n}(\varphi(x), f(x)), \quad a \in GL(m, K).$$

Denote

$$M_{n,m}(\mathcal{O}) = \{x \in M_{n,m} \mid x_{ij} \in \mathcal{O}\}, \quad U_m = \{u \in M_{n,m}(\mathcal{O}) \mid |\det u| = 1\}.$$

We shall call the homogeneous generalized function φ spherically symmetric if

$$(\varphi(x), f(ux)) = (\varphi(x), f(x)), \quad u \in U_n.$$

For $\lambda \neq -n + i$, $i = 0, 1, \dots, m-1$, the generalized function $D_{n,m}^\lambda$ is, obviously, a spherically symmetric homogeneous generalized function; for $\lambda = -n + i$, $i = 0, 1, \dots, m-1$, such is the residue of $D_{n,m}^\lambda$ at the corresponding points. Thus, for every complex number λ , we have constructed a spherically symmetric homogeneous generalized function of homogeneity degree λ . It is convenient to pass from the functions $D_{n,m}^\lambda(x)$ to the functions

$$F_\lambda(x) = \Phi(-\lambda - n, m) D_{n,m}^\lambda(x).$$

From what was said above it follows that $F_\lambda(x)$, as a function of λ , has neither singularities nor zeros in any finite domain. In particular, it is easy to see that

$$F_{-n}(x) = \Phi(-n, m) \delta(x). \quad (1)$$

Theorem 3. *For every complex number λ on $M_{n,m}$ there exists, up to a constant factor, only one generalized spherically symmetric homogeneous function of homogeneity degree λ , namely $F_\lambda(x)$.*

3. We define the Fourier transform of functions $f \in S_{n,m}$ by the formula

$$\tilde{f}(\xi) = \int_{M_{n,m}} f(x) \chi(\text{Sp}^t \xi \cdot x) dx,$$

* On the basis of Theorem 1, $D_{n,i}^\lambda$ is regular at $\lambda = -n + m$.

where $\chi(x)$ is an additive character of rank 0 (1) on K (the symbol t denotes transposition).

Let us note the obvious properties of the Fourier transform: 1) the function f is recovered from its Fourier transform $\tilde{f}$ by the formula

$$f(x) = \int_{M_{n,m}} \tilde{f}(\xi) \chi(-\text{Sp}^t \xi \cdot x) d\xi;$$

- 2) the Fourier transform is a bijective mapping of $S_{n,m}$ onto itself; 3) under the Fourier transform the characteristic function of the set $M_{n,m}(\mathcal{O})$ goes into itself. The Fourier transform $\tilde{\varphi}$ of a generalized function φ , as usual, is defined by the formula $(\tilde{\varphi}, \tilde{f}) = (\varphi, f(-x))$ for any $f \in S_{n,m}$.

Theorem 4. *The Fourier transform of the generalized function $F_\lambda(x)$ is given by the formula $\tilde{F}_\lambda(\xi) = F_{-\lambda-n}(\xi)$.*

4. Let us apply the result obtained to the solution of the following problem of integral geometry in the space $M_{n,m}$, $n > m$. We shall call a plane in $M_{n,m}$ a variety given by the linear matrix equation

$$(\xi, x) = \xi \cdot x = s, \quad (2)$$

where ξ is an $m \times n$ -matrix of maximal rank. Clearly, the dimension of such a plane is $m(n-m)$; the dimension of the variety of all planes of the form (2) is mn , i.e. it has the same dimension as $M_{n,m}$. To each function $f \in S_{n,m}$ we put in correspondence its integrals over all possible planes (2) by the formula

$$\check{f}(\xi, s) = \int_{M_{n,m}} f(x) \delta(s - \xi \cdot x) dx, \quad (3)$$

where δ is the delta function on $M_{m,m} = M_m \simeq K^{m^2}$, and dx is the volume element on $M_{n,m}$. The expression for $\check{f}(\xi, s)$ can be written directly in the form of an integral over the plane (2):

$$\check{f}(\xi, s) = \int_{\xi \cdot x = s} f(x) \omega_\xi,$$

where δ is the delta function on $M_{m,m} = M_m \simeq K^{m^2}$, dx is the volume element on $= dx^*$. It is not difficult to write an expression for ω_ξ in terms of the elements of the matrix x : $\omega_\xi = (-1)^{m(|\nu|-\rho)} |\det \xi_\nu|^{-m} dx^\nu$, where $\nu = (i^1, \dots, i^m)$ is an ordered set of numbers from the set $\{1, 2, \dots, n\}$; $|\nu| = i^1 + \dots + i^m$; ξ_ν is the matrix formed from the $i^1, \dots, i^m$ -th columns of the matrix ξ ; x^ν is the matrix obtained from the matrix x by deleting the $i^1, \dots, i^m$ -th rows; $\rho = m^2(n-mn+m^2+1)/2$.

We shall call the function $\check{f}(\xi, s)$ the Radon transform of the function $f(x)$. The problem consists in obtaining an inversion formula for the Radon transform (3). For what follows let us note one property of the Radon transform: if $f_{x_0}(x) = f(x + x_0)$, then $\check{f}_{x_0}(\xi, s) = \check{f}(\xi, s + \xi \cdot x_0)$.

5. Introduce the variety $U_{n,m} = \{u \in M_{n,m}(\mathcal{O}) \mid D_{n,m}(u) = 1\}$. It is easy to see that $U_{n,m}$, along with $M_{n,m}(\mathcal{O})$, is an open compact set in $M_{n,m}$.

Theorem 5. Let $\check{f}(\xi, s)$ be the Radon transform of a function $f \in S_{n,m}$. Then the inversion formula holds

$$f(x) = \frac{\Phi(n-m, m)}{\text{mes } U_m \Phi(-n, m)} \int_{U_{m,n}} \int_{M_m} \check{f}(u, s - u \cdot x) D_m^{-n}(s) ds du, \quad (4)$$

* That is, whatever the function $f \in S_{n,m}$, the measure ω_ξ is such that

$$\int_{M_m} ds \int_{\xi \cdot x = s} f(x) \omega_\xi = \int_{M_{n,m}} f(x) dx.$$

where $\text{mes } U_m = \Phi(-m, m)$, $du = \prod_{i,j}^{n,m} du_{ij}$ is a measure on the manifold $U_{n,m}$.

The proof is based on the equality

$$\int_{U_{m,n}} D_m^\lambda(u \cdot x) du = \frac{\text{mes } U_m \Phi(-\lambda - n, m)}{\Phi(-\lambda - m, m)} D_{n,m}^\lambda(x),$$

valid for all λ for which the integral converges. In view of this relation we have

$$\frac{\text{mes } U_m}{\Phi(-\lambda - m, m)} \int f(x) \Phi(-\lambda - n, m) D_{n,m}^\lambda(x) dx = \int f(x) D_m^\lambda(u \cdot x) du dx.$$

The integral on the right-hand side of this equality is easily expressed through the Radon transform, and we obtain

$$\frac{\text{mes } U_m}{\Phi(-\lambda - m, m)} \int f(x) \Phi(-\lambda - n, m) D_{n,m}^\lambda(x) dx = \int \check{f}(u, s) D_m^\lambda(s) ds du. \quad (5)$$

Both integrals converge for $\text{Re } \lambda > 0$ and in this domain are analytic functions of λ . Continue these functions to the domain $\text{Re } \lambda < 0$ and take their values at $\lambda = -n$. On the basis of (1), the value at $\lambda = -n$ of the function standing in the left-hand side of equality (5) is $\Phi(-n, m)f(0)$; D_m^λ is regular at $\lambda = -n$. Thus we have obtained

$$f(0) = \frac{\Phi(n-m, m)}{\text{mes } U_m \Phi(-n, m)} \int \check{f}(u, s) D_m^{-n}(s) ds du. \quad (6)$$

Now, to obtain the final formula (4), it remains only to apply formula (6) to the function $f_{x_0}(x) = f(x + x_0)$.

6. Theorem 6. In order that a function $\check{f}(\xi, s)$, $\xi \in M_{n,m} \setminus M_n^{m-1}$, be the Radon transform of some function $f \in S_{n,m}$, it is necessary and sufficient that the following conditions be satisfied: 1) $\check{f}(\xi, s)$ is a homogeneous function of ξ and s of degree of homogeneity $-m$, i.e. $\check{f}(a\xi, as) = |\det a|^{-m} \check{f}(\xi, s)$, $\forall a \in GL(m, K)$; 2) $\check{f}(\xi, s) = 0$ when $|s| |\xi|^{-1}$ is sufficiently large; 3) $\check{f}(\xi, s)$ is piecewise constant in the aggregate ξ, s , i.e., for any fixed ξ_0, s_0 we have $\check{f}(\xi_0, s_0) = \check{f}(\xi, s)$ when $|\xi - \xi_0|$ and $|s - s_0|$ are sufficiently small; 4) the integral $\int \check{f}(\xi, s) ds_i$, where $ds_i = ds_{i1} \dots ds_{im}$, does not depend on the elements of the i -th row of the matrix ξ .

The author takes this opportunity to express sincere gratitude to Prof. M. I. Graev for his help and valuable advice.

Kolomna Pedagogical
Institute

Received
4 VIII 1969

CITED LITERATURE

¹ I. M. Gel' fand, M. I. Graev, I. I. Pyatetskii-Shapiro, *Generalized Functions*, vol. 6, "Nauka," 1966.

² E. E. Petrov, DAN, **177**, No. 4, 782 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.