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Abstract

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THEORY OF ELASTICITY

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SPECTRUM OF A SYSTEM OF MEMBRANE EQUATIONS IN THE CASE OF AXISYMMETRIC VIBRATIONS OF A SHELL OF REVOLUTION

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1. The system of equations for axisymmetric vibrations of a shell of revolution has the form

$$-\frac{d}{ds} \left(\frac{1}{B(s)} \frac{dB(s)u}{ds} \right) - \frac{1-\sigma}{R_1(s)R_2(s)} u + \left(\frac{1}{R_1} + \frac{\sigma}{R_2} \right) \frac{dw}{ds} + \frac{d}{ds} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) w = \lambda u, \quad (1)$$

$$\frac{h^2}{12} \frac{1}{B} \frac{d}{ds} \left(B \frac{d}{ds} \left(\frac{1}{B} \frac{d}{ds} \left(B \frac{dw}{ds} \right) \right) \right) - \left(\frac{1}{R_1} + \frac{\sigma}{R_2} \right) \frac{du}{ds} - \left(\frac{\sigma}{R_1} + \frac{1}{R_2} \right) \frac{B'_s}{B} u + \left(\frac{1}{R_1^2} + \frac{2\sigma}{R_1 R_2} + \frac{1}{R_2^2} \right) w = \lambda w. \quad (1')$$

Here s is the length of the meridian arc, $a \leq s \leq b$; $B(s)$ is the abscissa of the meridian of the middle surface, and $R_1^{-1}(s)$ and $R_2^{-1}(s)$ are the principal curvatures of the surface of revolution:

$$R_1^{-1}(s) = -B''_{ss}(1 - B'^2_s)^{-1/2}; \quad R_2^{-1}(s) = (1 - B'^2_s)^{1/2} B^{-1}; \quad (2)$$

σ is Poisson's ratio; λ is the spectral parameter, equal to $(1 - \sigma^2)p^2 \rho E^{-1}$, where ρ is the density, E is Young's modulus, and p is the vibration frequency.

The functions $u(s)$ and $w(s)$ are the components of the displacement vector along the meridian and along the normal to the surface; h is a small parameter.

In the case of rigid clamping of the shell edges, the boundary conditions have the form

$$u(a) = u(b) = w(a) = w(b) = w'_s(a) = w'_s(b) = 0 \quad (3)$$

(see, in this connection, $(^{1,2,5})$).

To find the zero approximations of the frequencies of problem (1), (3) with respect to the small parameter h , one sets $h = 0$ in (1'). The resulting system of differential equations of second order will be written briefly in the form

$$L_0 f = \lambda f, \quad (4)$$

where $f(s) = (u(s), w(s))$ is a vector function, and we shall study the spectrum of system (4) under the boundary conditions

$$u(a) = u(b) = 0. \quad (4')$$

It can be shown that the operator L_0 , in the space of smooth vector functions $f(s)$ satisfying the boundary conditions (4'), is symmetric and positive definite. It is assumed that the scalar product is introduced by the formula

$$(f_1, f_2) = \int_a^b B(s)(u_1(s)\bar{u}_2(s) + w_1(s)\bar{w}_2(s)) ds. \quad (5)$$

The closure of this operator* will be denoted by L_0 .

2. We rewrite system (3) in understandable notation

$$L_{11}u + L_{12}w = \lambda u, \quad (6)$$

$$L_{21}u + L_{22}w = \lambda w \quad (6')$$

and introduce the functions

$$\varphi_1(s) = \frac{1 - \sigma^2}{R_2^2(s)}; \quad \varphi_2(s) = \frac{1}{R_1^2(s)} + \frac{2\sigma}{R_1(s)R_2(s)} + \frac{1}{R_2^2(s)}. \quad (7)$$

It is not difficult to verify that $\varphi_2(s) \geq \varphi_1(s)$ ($s \in [a, b]$). Let us also denote the set of values of the function $\varphi_1(s)$ by $[\alpha, \beta]$, and the set of values of $\varphi_2(s)$ by $[\gamma, \delta]$. In what follows we shall assume that the function $B(s)$ (the abscissa of the meridian) is sufficiently smooth and positive.

We shall need the following auxiliary propositions.

Lemma 1. Let $s_0 \in [a, b]$, $\lambda \notin [\alpha, \beta]$, and $\lambda \neq \varphi_2(s_0)$. Let r_1 and r_2 be arbitrary real numbers. Then there exists a unique solution of system (4) $f(s, \lambda) = (u(s, \lambda), w(s, \lambda))$ satisfying the Cauchy conditions

$$u(s_0, \lambda) = r_1, \quad u'_s(s_0, \lambda) = r_2. \quad (8)$$

The vector function $f(s, \lambda)$, for any $s \in [a, b]$, is regular in the whole λ -plane with the segment $[\alpha, \beta]$ and the point $\lambda = \varphi_2(s_0)$, at which it has a simple pole, removed.

Lemma 1 is proved by reducing the Cauchy problem to the Volterra equation

$$(\varphi_1(s) - \lambda)w(s, \lambda) + \int_{s_0}^s K(s, t, \lambda)w(t, \lambda) dt = \tau(s, \lambda), \quad (9)$$

in which $K(s, t, \lambda)$ is an entire function of λ , while $\tau(s, \lambda)$ has a pole at $\lambda = \varphi_2(s_0)$.

Lemma 2. On the resolvent set of the operator L_{11}^{**} the spectrum of problem (4), (4') coincides with the spectrum of the equation

$$(\varphi_1(s) - \lambda)w + T(\lambda)w = T_1(\lambda)h_1 + h_2, \quad (10)$$

where $T(\lambda)$ and $T_1(\lambda)$ are meromorphic completely continuous operators with poles at the points of the spectrum of the operator L_{11} ; $h_1(s)$ and $h_2(s)$ are arbitrary functions from $\mathcal{L}_2(a, b)$.

Using Lemmas 1 and 2, the following is proved.

Theorem 1. The spectrum of problem (4), (4') for $\lambda \notin [\alpha, \beta]$ is discrete (consists of isolated eigenvalues of finite multiplicity). The eigenvalues have an accumulation point $\lambda = +\infty$ and, possibly, the points $\lambda = \alpha$ and $\lambda = \beta$. The entire segment $[\alpha, \beta]$ belongs to the spectrum of the operator L_0 . All eigenvalues not belonging to the segments $[\alpha, \beta]$ and $[\gamma, \delta]$ are simple; eigenvalues belonging to the segment $[\gamma, \delta]$ are at most double.

2. We shall now show that, for the eigenfunctions $f_k(s) = (u_k(s), w_k(s))$ of problem (4), (4'), oscillation theorems hold.

The following proposition is fundamental.

Theorem 2. Let $\lambda \notin [\alpha, \beta]$, $\lambda \notin [\gamma, \delta]$. Let $f_0(s, \lambda) = (u_0(s, \lambda), w_0(s, \lambda))$ be the solution of system (4) satisfying the condition $u_0(a, \lambda) = 0$,

* It is easy to show that the operator L_0 , defined on the manifold of smooth vector functions satisfying conditions (4'), is essentially self-adjoint.

** The operator L_{11} is defined by the formula

$$L_{11}u = -\frac{d}{ds} \left(\frac{1}{B} \frac{dBu}{ds} \right) - \frac{1-\sigma}{R_1 R_2} u$$

and acts on scalar functions $u(s)$ satisfying condition (4'). The spectrum of L_{11} , as is well known, is real and discrete. The resolvent set is the complement of the spectrum.

$u'_0(a, \lambda) = 1$. Let $s_0(\lambda)$ be a zero of the function $u_0(s, \lambda)$ belonging to the interval (a, b) . Then the formula

$$\frac{ds_0}{d\lambda} = -\frac{\lambda - \varphi_2(s_0)}{\lambda - \varphi_1(s_0)} \left(\frac{\partial u_0(s_0)}{\partial s} \right)^{-2} \left(\int_a^{s_0} B(u_0^2 + w_0^2) ds \right) B^{-1}(s_0), \quad (11)$$

is valid, and consequently, for $\lambda < \alpha$ and $\lambda > \delta$ all zeros of the function $u_0(s, \lambda)$ move to the left as λ increases, while for $\beta < \lambda < \gamma$ they move to the right.

We agree to call the eigenvalues belonging to the interval $(0, \alpha)$ the first series, those belonging to the interval (β, γ) the second series, those belonging to the segment $[\gamma, \delta]$ the third series, and, finally, the eigenvalues belonging to the interval $(\delta, +\infty)$ the fourth series. Note that, according to Theorem 1, the fourth series is infinite, the third is finite, and the first and second may be finite or infinite*. Within each series we renumber the eigenvalues in increasing order.

Let $n_1(\lambda)$ be the number of zeros of the function $u_0(s, \lambda)$ for $s \in (a, b)$, when $\lambda < \alpha$, and let

$$N_1 = \sup_{\lambda < \alpha} n_1(\lambda) \quad (N_1 \leq +\infty). \quad (12)$$

Theorem 2 allows us to make the following conclusion:

Corollary 1. *The first series consists of N_1 eigenvalues $\lambda_0^{(1)}, \lambda_1^{(1)}, \dots, \lambda_k^{(1)}, \dots$, and the component $u_0(s, \lambda_k^{(1)})$ corresponding to the eigenfunction has exactly k zeros inside the interval (a, b) .*

Analogous assertions are valid for the second and fourth series. In particular, the first component of the k -th eigenfunction of the fourth series has $k + M_3$ zeros on the interval (a, b) , where $M_3 = \inf_{\lambda > \delta} n_3(\lambda)$, and $n_3(\lambda)$ is the number of zeros of $u_0(s, \lambda)$ for $\lambda > \delta$. Note that the eigenfunctions of the third series may have any number of zeros.

Theorem 3. *Suppose that in equation (4) the function $1/R_1(s) + \sigma/R_2(s)$ (see (1)) preserves its sign on the segment $[a, b]$. Then between any two neighboring zeros of the component $u(s, \lambda_k^{(i)})$ of the eigenfunction $f_k^{(i)}(s)$ of the first, second, and fourth series there lies strictly one zero of the component $w(s, \lambda_k^{(i)})$, $i = 1, 2, 4$. Thus, the zeros of the components of the eigenfunctions $f_k^{(i)}(s)$ strictly interlace.*

4. We now give a sufficient condition under which the first series of frequencies (the lowest) is infinite.

Theorem 4. *Suppose that in a neighborhood of the point $s = s_0 \in [a, b]$ the function $B(s)$ attains a maximum, so that $B(s) = B_0 - \frac{1}{2}k(s - s_0)^2 + O[(s - s_0)^3]$. Suppose $0 \leq B_0 k \leq 1$, and the local minimum attained by $\varphi_1(s)$ at the point s_0 coincides with its infimum $\varphi_1(s_0) = \alpha > 0$. Then, under the condition*

$$9(kB_0)^2 + (12\sigma - 1)kB_0 + 4\sigma^2 > 0 \quad (13)$$

the first series is infinite. For $\sigma > 1/24$, all kB_0 satisfy this condition.

The proof is carried out by studying the zeros of the solution $u_0(s, \lambda)$ and is based on Corollary 1 of Theorem 2.

In conclusion, note that in the case of a spherical shell the first series is infinite (this fact has already been noted in the literature, see ^(3,4)), while in the case of a conical shell it is, generally speaking, not empty. The number of eigenvalues in the first series can be made arbitrarily large; for this it is sufficient to take a cone with the angle of inclination of the generator close to $\pi/2$.

* Of course, the first, second, and third series may be absent altogether.

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Note: Figure translations are in progress. See original paper for figures.

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