

SOLUTION OF THE DIRICHLET PROBLEM FOR A HALF-SPACE

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Abstract

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MATHEMATICS

S. B. TOPURIA

SOLUTION OF THE DIRICHLET PROBLEM FOR A HALF-SPACE

(Presented by Academician I. N. Vekua on 27 IV 1970)

In the present note we state theorems which are, in a certain sense, analogues of Fatou's theorem for the Poisson integral in space, and on the basis of these theorems we solve the Dirichlet problem for a half-space in the formulation of N. N. Luzin ((1), p. 85). This problem for a ball was considered in (2).

1. Notation and definitions. In this paper the following notation is adopted: $R = (-\infty < x < \infty; -\infty < y < \infty)$; $\omega(P; h)$ —the circle of radius h with center at the point $P \in R$; $\tilde{L}(R)$ —the set of functions $f(x, y)$ such that the function $f(x, y)/(1 + x^2 + y^2)^{3/2}$ is integrable on R ; Z^+ —the set of points (x, y, z) for which $z > 0$; $\Delta_{xy} = \partial^2/\partial x^2 + \partial^2/\partial y^2$; $U(f; x, y, z)$ —the Poisson integral for the function $f(x, y)$ in the space Z^+ , i.e.

$$U(f; x, y, z) = \frac{z}{2\pi} \iint_R \frac{f(t, \tau) dt d\tau}{[(t-x)^2 + (\tau-y)^2 + z^2]^{3/2}};$$

the symbol $(h, k)_\lambda \rightarrow 0$ means that $h \rightarrow 0$, $k \rightarrow 0$, and

$$1/\lambda \leq |h/k| \leq \lambda, \quad \lambda \geq 1; \quad M(x, y, z) \xrightarrow{\Lambda} P(x_0, y_0, 0)$$

means that the point M tends to P along paths non-tangential to R , i.e. there exists a positive number k such that

$$z/\sqrt{(x-x_0)^2 + (y-y_0)^2} \geq k.$$

Definitions. 1) The generalized Laplace operator is defined by the formula (3, 4):

$$\Delta^* f(P) = \lim_{h \rightarrow 0} \frac{8}{\pi h^4} \iint_{\omega(P; h)} [f(Q) - f(P)] d\omega_Q;$$

- 2) The derivatives $Df(x, y)$, $\widetilde{D}f(x, y)$, $D_\lambda f(x, y)$, $C_1 Df(x, y)$, $C_1 D^* f(x, y)$, and $C_{1\lambda} Df(x, y)$ of the function $f(x, y)$ at the point $P(x, y)$ are defined as follows:

$$Df(x, y) = \lim_{h, k \rightarrow 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk};$$

$$\widetilde{D}f(x, y) =$$

$$\lim_{h, k \rightarrow 0} \frac{f(x+h, y+k) + f(x-h, y+k) + f(x+h, y-k) + f(x-h, y-k) - 4f(x, y)}{h^2 + k^2};$$

$$D_\lambda f(x, y) = \lim_{(h, k)_\lambda \rightarrow 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk};$$

$$C_1 Df(x, y) = \lim_{h, k \rightarrow 0} \frac{4}{h^2 k^2} \int_x^{x+h} \int_y^{y+k} [f(t, \tau) - f(t, y) - f(x, \tau) + f(x, y)] dt d\tau;$$

$$C_1 D^* f(x, y) = \lim_{h, k \rightarrow 0} \frac{1}{h^2 k^2} \int_0^h \int_0^k [f(x+t, y+\tau) - f(x-t, y+\tau) - f(x+t, y-\tau) + f(x-t, y-\tau)] dt d\tau;$$

$$C_{1\lambda} Df(x, y) = \lim_{(h, k) \xrightarrow{\Lambda} 0} \frac{4}{h^2 k^2} \int_x^{x+h} \int_y^{y+k} [f(t, \tau) - f(t, y) - f(x, \tau) + f(x, y)] dt d\tau.$$

2. Fatou theorems for the Poisson integral in the space Z^+

The Fatou theorem for the Poisson integral in the space Z^+ has certain special features. The behavior of the differentiated Poisson integral depends essentially on the sense in which the density of the integral is differentiated.

The following theorems hold:

Theorem 1. a) If $f(x, y) \in \widetilde{L}(R)$ and at the point $P(x, y, 0)$ there exists a finite derivative $\Delta^* f(x, y)$, then

$$\lim_{z \rightarrow 0+} \Delta_{xy} U(f; x, y, z) = \Delta^* f(x, y).$$

b) There exists a continuous function $f(x, y)$ such that $\widetilde{D}f(0, 0) = 0$, but

$$\lim_{\substack{(x, y, z) \rightarrow (0, 0, 0) \\ \Lambda}} \Delta_{xy} U(f; x, y, z)$$

does not exist.

Theorem 2. If $f(x, y) \in \widetilde{L}(R)$ and at the point $P(x, y, 0)$ there exists a finite derivative $C_1 D^* f(x, y)$, then

$$\lim_{z \rightarrow 0+} \frac{\partial^2 U(f; x, y, z)}{\partial x \partial y} = C_1 D^* f(x, y).$$

Theorem 3. If $f(x, y) \in \widetilde{L}(R)$ and at the point $P(x_0, y_0, 0)$ there exists a finite derivative $C_1 Df(x_0, y_0)$, then

$$\lim_{\substack{(x, y, z) \rightarrow (x_0, y_0, 0) \\ \Lambda}} \frac{\partial^2 U(f; x, y, z)}{\partial x \partial y} = C_1 Df(x_0, y_0).$$

Theorem 4. There exists a continuous function $f(x, y)$ such that $D_\Lambda f(0, 0) = 0$, but

$$\lim_{z \rightarrow 0+} \frac{\partial^2 U(f; 0, 0, z)}{\partial x \partial y} = +\infty.$$

Theorem 5. There exists a continuous function $f(x, y)$ such that $C_{1\Lambda} Df(0, 0) = 0$, but

$$\lim_{z \rightarrow 0+} \frac{\partial^2 U(f; 0, 0, z)}{\partial x \partial y} = +\infty.$$

Theorem 6. a) If $f(x, y) \in \widetilde{L}(R)$ and $f(x, y)$, at the point $P(x_0, y_0, 0)$, has a total differential $df(x_0, y_0)$, then

$$\lim_{\substack{(x, y, z) \rightarrow (x_0, y_0, 0) \\ \Lambda}} \frac{\partial U(f; x, y, z)}{\partial x} = \frac{\partial f(x_0, y_0)}{\partial x};$$

$$\lim_{\substack{(x, y, z) \rightarrow (x_0, y_0, 0) \\ \Lambda}} \frac{\partial U(f; x, y, z)}{\partial y} = \frac{\partial f(x_0, y_0)}{\partial y}.$$

b) There exists a continuous function $f(x, y)$ such that $\partial f(0, 0)/\partial x = \partial f(0, 0)/\partial y = 0$, but

$$\lim_{z \rightarrow 0+} \frac{\partial U(f; 0, 0, z)}{\partial x} = +\infty.$$

3. The Dirichlet Problem for a Half-Space

The Dirichlet problem for the Laplace equation consists in determining a function $U(x, y, z)$ in the domain Z^+ with boundary R , satisfying the equation $\Delta U = 0$ and the boundary condition $U|_R = f(x, y)$. In paper ⁵, a solution of this problem is given in the case when $f(x, y) \in \tilde{L}(R)$. In the present note this problem is solved for the case when the boundary function is measurable and finite almost everywhere on R , i.e., in the formulation of N. N. Luzin.

Theorem 7. Let $f(x, y)$ be an arbitrary measurable function, finite almost everywhere on R . Then there exists a bounded continuous function $F(x, y)$ such that almost everywhere on R

$$DF(x, y) = f(x, y).$$

Theorem 8. Let $f(x, y)$ be an arbitrary measurable function, finite almost everywhere on R . Then there exists a harmonic function $U(x, y, z)$ in Z^+ such that

$$\lim_{(x, y, z) \rightarrow (x_0, y_0, 0)} U(x, y, z) = f(x_0, y_0)$$

almost everywhere on R .

All the theorems stated above are valid for any n -dimensional Euclidean space.

Georgian Polytechnic Institute
named after V. I. Lenin
Tbilisi

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Note: Figure translations are in progress. See original paper for figures.

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