

**THE HILBERT  
PARALLELEPIPED  
CANNOT BE  
DECOMPOSED INTO A  
COUNTABLE UNION  
OF CLOSED SUBSETS  
DISTINCT FROM IT,  
WHOSE PAIRWISE  
INTERSECTIONS ARE  
WEAKLY INFINITE-  
DIMENSIONAL**

N. Hadzhiivanov

1970

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.16883>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 513.83

**MATHEMATICS**

**N. HADJIIVANOV (BULGARIA)**

# THE HILBERT PARALLELEPIPED CANNOT BE DECOMPOSED INTO A COUNTABLE UNION OF CLOSED SUBSETS DISTINCT FROM IT, WHOSE PAIRWISE INTERSECTIONS ARE WEAKLY INFINITE-DIMENSIONAL

*(Presented by Academician P. S. Aleksandrov, 20 V 1970)*

The aim of the present note is to prove the assertion announced in the title.

We shall call a space  $X$  **weakly infinite-dimensional** if, for every countable system of pairs of closed sets  $A_{+n}$  and  $A_{-n}$  such that  $A_{+n} \cap A_{-n} = \emptyset$ , one can choose partitions\*  $C_n$  between  $A_{+n}$  and  $A_{-n}$  whose intersection is empty:

$$\bigcap_{n=1}^{\infty} C_n = \emptyset \quad (1)$$

Obviously, a closed subset of a weakly infinite-dimensional space will also be weakly infinite-dimensional.

The following two lemmas require the concept of normal adjoining, introduced for another purpose by Yu. M. Smirnov (see <sup>(2)</sup>):

A set  $N$  of a topological space  $X$  **normally adjoins** its complement  $M = X \setminus N$  if any two disjoint closed subsets of  $M$  have disjoint neighborhoods open in  $X$ .

**Lemma 1.** *Every subset of type  $G_\delta$  of a normal space normally adjoins its complement.*

We omit the simple proof of this lemma.

**Lemma 2.** *If, in a normal space  $X$ , the subset  $X \setminus M$  normally adjoins the weakly infinite-dimensional countably paracompact subset  $M$ , then for every countable system of pairs of closed sets  $(A_{+n}, A_{-n})$  such that  $A_{+n} \cap A_{-n} = \emptyset$ , one can choose partitions  $C_n$  between  $A_{+n}$  and  $A_{-n}$ , the intersection of which has no common points with  $M$ :*

$$M : \quad M \cap \bigcap_{n=1}^{\infty} C_n = \emptyset.$$

For any pair  $(A_{+n}, A_{-n})$  take neighborhoods  $U_{+n}$  and  $U_{-n}$ , open in  $X$ , such that  $\overline{U_{+n}} \cap \overline{U_{-n}} = \emptyset$ . For the countable system of pairs of closed subsets in  $M$ ,  $(M \cap \overline{U_{+n}}, M \cap \overline{U_{-n}})$ , one can choose partitions  $C'_n$  in  $M$  between  $M \cap \overline{U_{+n}}$  and  $M \cap \overline{U_{-n}}$ , whose intersection is empty:

$$\bigcap_{n=1}^{\infty} C'_n = \emptyset.$$

In the countably paracompact space  $M$ , the partitions  $C'_n$  can be enlarged to such open subsets  $L_n$  of  $M$  that

$$\bigcap_{n=1}^{\infty} L_n = \emptyset,$$

and  $L_n$  does not intersect  $M \cap \overline{U_{+n}}$  and  $M \cap \overline{U_{-n}}$  for any  $n$ . The sets  $C'_n$  are partitions between  $M \cap \overline{U_{+n}}$  and  $M \cap \overline{U_{-n}}$ , and, consequently,  $M \setminus C'_n = M_{+n} \cup M_{-n}$ , where  $M_{+n}$  and  $M_{-n}$  are disjoint neighborhoods, open in  $M$ , of the sets  $M \cap \overline{U_{+n}}$  and  $M \cap \overline{U_{-n}}$ . Since the sets closed in  $M$ ,  $M_{+n} \setminus L_n$  and  $M_{-n} \setminus L_n$ , do not pe-

\* A closed subset  $C$  of a space  $X$  is called a partition between the sets  $A$  and  $B$  if the complement  $X \setminus C$  is decomposed into such disjoint open sets  $G$  and  $H$  that  $A \subset G$ ,  $B \subset H$ .

intersect, they have disjoint open neighborhoods  $V_{+n}$  and  $V_{-n}$  in  $X$ . The sets

$$C_n = X \setminus (U_{+n} \cup (V_{+n} \setminus \overline{U_{-n}}) \cup U_{-n} \cup (V_{-n} \setminus \overline{U_{+n}}))$$

are partitions between  $A_{+n}$  and  $A_{-n}$  in the space  $X$ , and

$$M \cap \bigcap_{n=1}^{\infty} C_n = \emptyset.$$

**Lemma 3.** *If a normal space  $X$  is the union of its weakly infinite-dimensional countably paracompact subsets  $X_i$ , where  $X \setminus X_i$  is normally situated with respect to  $X_i$ ,  $i = 1, 2, \dots$ , then the space  $X$  is weakly infinite-dimensional.*

Let, in the infinite matrix

$$\begin{aligned} n_{11}, n_{12}, n_{13}, \dots \\ n_{21}, n_{22}, n_{23}, \dots \\ n_{31}, n_{32}, n_{33}, \dots \\ \dots \dots \dots \end{aligned}$$

each natural number occur exactly once. If  $(A_{+n}, A_{-n})$ ,  $n = 1, 2, \dots$ , is a countable system of pairs of closed disjoint sets in  $X$ , then we must construct partitions  $C_n$  in  $X$  between  $A_{+n}$  and  $A_{-n}$  such that

$$\bigcap_{n=1}^{\infty} C_n = \emptyset.$$

By Lemma 2, for the countable system of pairs  $(A_{+n_{ki}}, A_{-n_{ki}})$ ,  $i = 1, 2, \dots$ , one can choose partitions  $C_{n_{ki}}$  in  $X$  between  $A_{+n_{ki}}$  and  $A_{-n_{ki}}$  such that

$$X_k \cap \bigcap_{i=1}^{\infty} C_{n_{ki}} = \emptyset.$$

Then  $\{C_{n_{ki}}\}_{k,i=1}^{\infty}$  will be the required system of partitions.

In the Hilbert parallelepiped

$$I_{\aleph_0}^{\aleph_0} = [-1, 1]_{\aleph_0}^{\aleph_0}$$

any subset

$$P = \prod_{n=1}^{\infty} [\alpha_n, \beta_n]$$

will also be called a parallelepiped, and of its subsets

$$P_{-i} = \alpha_i \times \prod_{n \neq i} [\alpha_n, \beta_n] \quad \text{and} \quad P_{+i} = \beta_i \times \prod_{n \neq i} [\alpha_n, \beta_n]$$

we shall say that they are its opposite faces.

Let

$$P \subset \bigcup_{i=1}^{\infty} \Phi_i,$$

where  $\Phi_i$  are closed subsets of the space  $I^{\aleph_0}$ ,  $P \not\subset \Phi_i$  for no  $i$ , and the intersections  $\Phi_i \cap \Phi_j$  are weakly infinite-dimensional for any distinct  $i$  and  $j$ . Note that the set  $X \setminus M$  is normally situated with respect to

$$M = \bigcup_{i \neq j} (\Phi_i \cap \Phi_j)$$

(see Lemma 1), and the set  $M$  is weakly infinite-dimensional (see Lemma 3).

**Lemma 4.** *For any  $i$ , at least one of the summands  $\Phi_k$  contains a continuum joining the opposite faces  $P_{-i}$  and  $P_{+i}$  of the parallelepiped  $P$ .*

By symmetry it suffices to restrict ourselves to the case  $i = 1$ . By Lemma 2 there exist partitions in  $P$  between the sets  $P_{-n}$  and  $P_{+n}$ ,  $n = 2, 3, \dots$ , such that the set

$$C = \bigcap_{n=2}^{\infty} C_n$$

does not meet the set  $M$ . To prove the lemma, it is enough to find at least one component of connectedness  $K$  of the compactum  $C$  joining  $P_{-1}$  with  $P_{+1}$ . Indeed, since  $K \cap M = \emptyset$ , the continuum  $K$  is decomposed into the union of pairwise disjoint closed sets  $K \cap \Phi_j$ . Hence, by a theorem of Sierpiński <sup>(3)</sup>,

$$K = K \cap \Phi_k$$

for some natural  $k$ , i.e.  $K \subset \Phi_k$ , which is what the lemma requires.

We shall show that such a component can be found. Indeed, if there were no such components, then it is not difficult to show that the compactum  $C$  can be split into the sum of disjoint open-and-closed-in- $C$  sets  $C_+$  and

$C_-$  in such a way that  $C_+ \cap P_{-1} = \emptyset$  and  $C_- \cap P_{+1} = \emptyset$ . It is clear that the closed sets  $P_{+1} \cup C_+$  and  $P_{-1} \cup C_-$  do not intersect. Hence they can be separated in the parallelepiped  $P$  by a closed partition  $C_1$ . Thus the opposite faces  $P_{-n}$  and  $P_{+n}$ ,  $n = 1, 2, \dots$ , of the parallelepiped  $P$  have turned out to be separable by partitions  $C_n$  with empty intersection

$$\bigcap_{n=1}^{\infty} C_n = \emptyset,$$

which is impossible (see (1), p. 75). Consequently, contrary to the assumption, there exists a component of connectedness  $K$  of the compactum  $C$  joining the face  $P_{-1}$  with the face  $P_{+1}$ , as was to be proved.

**Lemma 5.** *No summand  $\Phi_k$  contains any parallelepiped  $N$  two of whose faces are parallel to the faces  $P_{-i}$  and  $P_{+i}$ , while the remaining ones lie on the faces of the parallelepiped  $P$ , i.e.*

$$\Phi_k \supset N[a_i, b_i] \times \prod_{n \neq i} (\alpha_n, \beta_n),$$

where  $[a_i, b_i] \subset [\alpha_i, \beta_i]$ .

Of course, we may again assume that  $i = 1$ . Suppose that, for example, the summand  $\Phi_1$  contains entirely some parallelepiped  $N$  of the indicated form. Then the union of the closed differences  $\Phi_k \setminus N^{\circ*}$ ,  $k \neq 1$ , together with the first summand  $\Phi_1$ , covers the parallelepiped  $P$ ; none of them contains  $P$ , and all their pairwise intersections are weakly infinite-dimensional. Therefore we may assume that the given decomposition is such that some parallelepiped of the indicated form does not intersect any of the summands  $\Phi_k$ ,  $k \neq 1$ . Since the closed set  $\Phi_1$  does not contain  $P$ , there will be found in  $P$  a parallelepiped

$$Q = \prod_{k=1}^{\infty} [\gamma_k, \delta_k],$$

where  $[\gamma_k, \delta_k] = [\alpha_k, \beta_k]$  for  $k \neq k_i$ ,  $i = 1, 2, \dots, m$ , and  $Q \cap \Phi_1 = \emptyset$ . The parallelepiped

$$R = [\alpha_1, \beta_1] \times \prod_{n=2}^{\infty} [\gamma_n, \delta_n],$$

obviously contains  $Q$ .  $R$  does not lie in any of the summands  $\Phi_k$ ,  $k \neq 1$ , since otherwise  $N$  would intersect one of them. Nor does  $R$  lie in  $\Phi_1$ , since  $Q$  does not lie in  $\Phi_1$ . Hence, by Lemma 4, in  $R$  there exists a continuum  $K$  lying in some  $\Phi_k$  and joining the faces of the parallelepiped  $R$  that lie on the faces  $P_{-1}$  and  $P_{+1}$ . This continuum  $K$  necessarily intersects  $N$  and, consequently,  $\Phi_k \cap N \neq \emptyset$ , which entails  $k = 1$ , since  $K \subset \Phi_1$ . But the continuum  $K$  also intersects  $Q$ , so that  $Q \cap \Phi_1 \neq \emptyset$ , contradicting the choice of the parallelepiped  $Q$ .

We proceed to the proof of the proposition formulated in the title.

Let

$$I^{\aleph_0} = \bigcup_{k=1}^{\infty} \Phi_k,$$

where the closed subsets  $\Phi_k$  do not coincide with  $I^{\aleph_0}$  and the intersections  $\Phi_i \cap \Phi_j$  are weakly infinite-dimensional for  $i \neq j$ . Since  $I^{\aleph_0}$  is a complete metric

space, all the summands  $\Phi_k$  cannot be nowhere dense. Hence at least one of the summands, let it be  $\Phi_1$ , contains entirely some parallelepiped

$$Q = \prod_{k=1}^{\infty} [a_k, b_k],$$

where for  $k > n$  we have  $[a_k, b_k] = [-1, 1]$ , and  $-1 \leq a_k < b_k \leq 1$  for  $k = 1, 2, \dots, n$ . Put

$$P^{(i)} = \prod_{k=1}^{\infty} [a_k^{(i)}, b_k^{(i)}], \quad i = 1, 2, \dots, n+1,$$

where  $[a_k^{(i)}, b_k^{(i)}] = [-1, 1]$  for  $k \geq i$ , and  $[a_k^{(i)}, b_k^{(i)}] = [a_k, b_k]$  for  $k < i$ . It is clear that  $P^{(1)} = I^{\aleph_0}$ ,  $P^{(n+1)} = Q$  and

$$P^{(i+1)} = [a_i, b_i] \times \prod_{k \neq i} [a_k^{(i)}, b_k^{(i)}].$$

By Lemma 5 the parallelepiped  $P^{(2)}$  is not contained in any of the summands  $\Phi_k$ . Again by Lemma 5 we conclude that  $P^{(3)}$  is not contained entirely in any of the summands  $\Phi_k$ . Continuing these

---


$$* N = (a_i, b_i) \times \prod_{n \neq i} [\alpha_n, \beta_n].$$

reasoning, at the  $n$ -th step we obtain that  $Q$  is not contained in any of the summands  $\Phi_k$ , which contradicts the choice of the parallelepiped  $Q$ . The proposition is proved.

With the aid of the assertion just proved it is not difficult to prove the following theorem.

**Theorem.** *If  $M$  is a connected topological space, every point  $x$  of which has a neighborhood homeomorphic to the product  $R^{\aleph_0}$ , where  $R$  is the real line, then  $M$  cannot be decomposed into a countable union of closed subsets, different from all of  $M$ , whose pairwise intersections are weakly infinite-dimensional.*

Let

$$M = \bigcup_{k=1}^{\infty} \Phi_k,$$

where the sets  $\Phi_k$  are closed,  $\Phi_k \neq M$  for no  $k$ , and the intersection  $\Phi_i \cap \Phi_j$  is weakly infinite-dimensional whenever  $i \neq j$ . Let  $x \in M$ , and let  $Ox$  be a neighborhood of the point  $x$  homeomorphic to the space  $R^{\aleph_0}$ ; for simplicity we

shall assume that  $Ox = R^{\aleph_0}$ . We shall prove that  $Ox$  necessarily lies in one of the summands  $\Phi_k$ .

Indeed,

$$Ox \subset \bigcup_{n=1}^{\infty} [-n, n]^{\aleph_0},$$

and since  $[-n, n]^{\aleph_0}$ , by what has already been proved, has no decomposition of the indicated kind, there exists a summand  $\Phi_{k_n}$  not containing the set  $[-n, n]^{\aleph_0}$ . We may assert that for every  $n$  we have  $k_n = k_1$ ; otherwise, from the inequality

$$[-1, 1]^{\aleph_0} \subset \Phi_{k_1} \cap \Phi_{k_n}$$

it would follow that the Hilbert cube  $[-1, 1]^{\aleph_0}$  is weakly infinite-dimensional, which is a contradiction (see (1), p. 75). Hence  $k_n = k_1$  for every  $n$ , and consequently  $Ox \subset \Phi_{k_1}$ . Thus  $U = \text{Int } \Phi_{k_1} \neq \emptyset$ . Since the space  $M$  is connected,  $\text{Fr } U \neq \emptyset$ . Now let  $y \in \text{Fr } U$ , and let  $Oy$  be a neighborhood of the point  $y$  homeomorphic to the space  $R^{\aleph_0}$ ; for simplicity we shall assume that  $Oy = R^{\aleph_0}$ . We already know that there exists a summand  $\Phi_{k_2}$  containing  $Oy$ . It is clear that  $k_2 \neq k_1$ , since otherwise we would have  $Oy \subset \Phi_{k_1}$ , i.e.  $y \in U$ , which contradicts the choice of the point  $y$ . There exists an index  $n$  such that

$$[-n, n]^{\aleph_0} \cap U \neq \emptyset \quad \text{and} \quad [-n, n]^{\aleph_0} \not\subset \Phi_{k_1}.$$

Indeed, if  $[-n, n]^{\aleph_0} \subset M \setminus U$  for every  $n$ , then

$$Oy \subset \bigcup_{n=1}^{\infty} [-n, n]^{\aleph_0} \subset M \setminus U,$$

and this entails  $Oy \cap U = \emptyset$ , which is a contradiction. On the other hand, if  $[-n, n]^{\aleph_0} \subset \Phi_{k_1}$  for every  $n$ , then

$$Oy \subset \bigcup_{n=1}^{\infty} [-n, n]^{\aleph_0} \subset \Phi_{k_1},$$

which is a contradiction. Put

$$F_1 = [-n, n]^{\aleph_0} \cap \Phi_{k_1}, \quad F_2 = [-n, n]^{\aleph_0} \setminus U.$$

It is clear that

$$F_1 \cup F_2 = [-n, n]^{\aleph_0}, \quad F_1 \neq [-n, n]^{\aleph_0}, \quad F_2 \neq [-n, n]^{\aleph_0}.$$

Moreover, the intersection  $F_1 \cap F_2$  is weakly infinite-dimensional, since it is a closed subset of the weakly infinite-dimensional set  $\Phi_{k_1} \cap \Phi_{k_2}$ . This contradicts the assertion formulated in the title. Consequently, a decomposition of the space of the indicated kind is impossible.

Faculty of Mechanics and Mathematics  
of Moscow State University  
named after M. V. Lomonosov

Faculty of Mathematics  
of Sofia University

Received  
12 V 1970

## REFERENCES

1. V. Hurewicz, H. Wallman, *Dimension Theory*, Moscow, 1948.
2. Yu. M. Smirnov, *Matem. sbornik*, **69**, 141 (1966).
3. W. Sierpinski, *Tôhoku Math. J.*, **13**, 300 (1918).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*