

On the Divergence of Fourier Series

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Abstract

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MATHEMATICS

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ON THE DIVERGENCE OF FOURIER SERIES

(Presented by Academician I. N. Vekua on 18 III 1970)

1. It is well known (see ⁽¹⁾ or ⁽²⁾, p. 599) that there exists a 2π -periodic function $f(x) \in L(-\pi, \pi)$, whose Fourier series $\sigma[f]$ does not converge in the metric $L(-\pi, \pi)$. On the other hand (see ⁽³⁾, p. 424), if $f(x) \in L \log^+ L$, then the series $\sigma[f]$ converges in the metric $L(-\pi, \pi)$. Naturally the question arises: what can be said about the convergence of the series $\sigma[f]$ in the sense of the space $L(-\pi, \pi)$ in the case when $f(x) \in L(\log^+ L)^\alpha$ for all $\alpha \in (0, 1)$?

Further, it is known (see ⁽⁴⁾, p. 258) that if $f(x) \in L(-\pi, \pi)$ and its integral modulus of continuity $\omega(\delta, f)_L$ satisfies the condition

$$\omega(\delta, f)_L = O \left\{ \left(\log \frac{1}{\delta} \right)^{-1-\varepsilon} \right\}, \quad \varepsilon > 0 \quad (\delta \rightarrow 0), \quad (1)$$

then the series $\sigma[f]$ converges both in the metric $L(-\pi, \pi)$ and almost everywhere. Whether condition (1) with $\varepsilon = 0$ guarantees the convergence of the series $\sigma[f]$ almost everywhere or in the sense of the metric $L(-\pi, \pi)$ is unknown. However (see ⁽³⁾, p. 288, or ⁽⁵⁾, p. 100), if

$$\omega(\delta, f)_L = o \left\{ \left(\log \frac{1}{\delta} \right)^{-1} \right\} \quad (\delta \rightarrow 0), \quad (2)$$

then the series $\sigma[f]$ converges in the metric $L(-\pi, \pi)$. It is unknown whether, in relation (2), the symbol o can be replaced by O .

Below we shall give assertions that answer the questions posed (with respect to convergence in the metric $L(-\pi, \pi)$); results will also be given that are generalizations of the assertions obtained by us to the case of multiple trigonometric Fourier series.

2. Let $f(x) \in L(-\pi, \pi)$. As usual, by $\bar{f}(x)$ we denote the conjugate function to $f(x)$, and by the symbol $\bar{\sigma}[f]$ the conjugate trigonometric series to $\sigma[f]$.

Theorem 1. There exists a function $f_0(x) \in L(-\pi, \pi)$ such that:

a)

$$\omega(\delta, f_0)_L = O\left\{\left(\log \frac{1}{\delta}\right)^{-1}\right\}, \quad \omega(\delta, \bar{f}_0)_L = O\left\{\left(\log \frac{1}{\delta}\right)^{-1}\right\} \quad (\delta \rightarrow 0);$$

b) the series $\sigma[f_0]$ and $\bar{\sigma}[f_0]$ do not converge in the metric $L(-\pi, \pi)$.

Theorem 1 shows that condition (2) is essential for the convergence of the series $\sigma[f]$ in the sense of the metric $L(-\pi, \pi)$; consequently, condition (1) with $\varepsilon = 0$, generally speaking, does not guarantee the convergence of the series $\sigma[f]$ in the metric $L(-\pi, \pi)$ (cf. the corresponding assertion from the book ⁽⁴⁾, p. 258).

We also note that in item b) of Theorem 1 one may restrict oneself to considering only one of the series $\sigma[f_0]$ and $\bar{\sigma}[f_0]$, since the functions $f_0(x)$ and $\bar{f}_0(x)$ are summable, and in this case (see ⁽²⁾, p. 602) the divergence of one series in the metric $L(-\pi, \pi)$ entails the divergence of the other series in the sense of the same metric.

Using the results of P. L. Ul'yanov ⁽⁶⁾, according to Theorem 1 we obtain

Corollary. There exists a function $f_0(x)$ such that $(f_0(x), \bar{f}_0(x)) \in L(\log^+ L)^\alpha$ for all $\alpha \in [0, 1)$, but the series $\sigma[f_0]$ and $\sigma[\bar{f}_0]$ do not converge in the metric $L(-\pi, \pi)$.

Theorem 1 shows that for the convergence of $\sigma[f]$ in the metric $L(-\pi, \pi)$ the condition $f(x) \in L \log^+ L$ is essential.

3. Let now $R = [-\pi, \pi; -\pi, \pi; \dots; -\pi, \pi]$, and consider a function $f(x_1, x_2, \dots, x_n) \in L(R)$ which is 2π -periodic with respect to each of the variables. By the symbols $\omega(\delta_1, \delta_2, \dots, \delta_n, f)_L, \omega(\delta_1, \delta_2, \dots, \delta_{n-1}, f)_L, \dots, \omega(\delta_1, \delta_2, f)_L, \dots, \omega(\delta_1, f)_L, \dots, \omega(\delta_1, \delta_2, \dots, \delta_n, f)_L$ we shall denote, respectively, the complete and partial integral moduli of continuity of the function $f(x_1, x_2, \dots, x_n)$, where it is assumed that δ_k corresponds to x_k ($k = 1, 2, \dots, n$). By $\sigma_n[f]$ we denote the n -fold (see (5), p. 256) trigonometric Fourier series of the function $f(x_1, x_2, \dots, x_n)$, and by the expressions $\bar{\sigma}_n[f, x_1], \bar{\sigma}_n[f, x_2], \dots, \bar{\sigma}_n[f, x_1, x_2], \dots, \bar{\sigma}_n[f, x_{n-1}, x_n], \dots, \bar{\sigma}_n[f, x_1, x_2, \dots, x_n]$ we shall denote the n -fold (see (5), p. 256) conjugate trigonometric series. As for (see (5), p. 257) the conjugate functions of the variables, we denote them by the symbols $\bar{f}_1(x_1, x_2, \dots, x_n), \bar{f}_2(x_1, x_2, \dots, x_n), \dots, \bar{f}_{1,2,\dots,n}(x_1, x_2, \dots, x_n)$.

Since, in the study of multiple series, one may consider different types of convergence, we shall restrict ourselves to considering convergence in the sense of Pringsheim.

Theorem 2. Let $f(x_1, x_2, \dots, x_n) \in L(R)$ and $1 \leq i \leq m$ ($m = 1, 2, \dots, n$). If

$$\omega(\delta_i, \delta_{i+1}, \dots, \delta_m, f)_L = o \left\{ \prod_{k=i}^m \left(\log \frac{1}{\delta_k} \right)^{-1} \right\} \quad (\delta_k \rightarrow 0, \quad k = 1, 2, \dots, n), \quad (3)$$

then the series $\sigma_n[f]$ converges in the metric $L(R)$.

The theorem just given is, in a certain sense, final, since the following is true.

Theorem 3. There exists a function $f_1(x_1, x_2, \dots, x_n) \in L(R)$ for which

$$\text{a) } \omega(\delta_i, \delta_{i+1}, \dots, \delta_m, f_1)_L = O \left\{ \prod_{k=i}^m \left(\log \frac{1}{\delta_k} \right)^{-1} \right\} \quad (\delta_k \rightarrow 0); \quad (4)$$

b) the integral moduli of continuity of all conjugate functions for $f_1(x_1, x_2, \dots, x_n)$ also satisfy relations (4);

c) the series $\sigma_n[f_1]$ does not converge in the metric $L(R)$.

We note that even the fulfillment of condition (3) in the sense that, for some m_0 and i_0 (which may be equal), o is replaced by O , does not guarantee convergence in the metric $L(R)$ of the series $\sigma_n[f]$.

We also note that some series among the n -fold conjugate trigonometric series may be convergent in the metric $L(R)$, something that does not occur in the one-dimensional case.

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Note: Figure translations are in progress. See original paper for figures.

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