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Abstract

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MATHEMATICS

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DIFFERENTIAL GAMES WITH CONSTRAINTS ON PHASE STATES

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A differential game is considered in which the payoff is the time until the phase point $x[t]$ meets a certain set $\mathcal{M}$. It is assumed that the motion of the controlled system must satisfy the constraint $x[t] \in \mathcal{X}$. Formulations of game problems of this type are given, and the structure of optimal approximating strategies is described.

Let the motion of the system be described by the equation

$$dx/dt = f^{(1)}(t, x, u) + f^{(2)}(t, x, v), \quad x(t_0) = x_0, \quad (1)$$

where x is the phase vector of the system; u and v are the vectors of the players' control actions, constrained by

$$u \in \mathcal{U}, \quad v \in \mathcal{V}; \quad (2)$$

the sets $\mathcal{U}$ and $\mathcal{V}$ are bounded and closed; the continuous functions $f^{(1)}, f^{(2)}$ satisfy a Lipschitz condition with respect to x .

We shall use the concepts and notation from paper ⁽¹⁾, where a bibliography is also given. In particular, we shall use the following notation:

$$\mathcal{F}^{(1)}(t, x) = \text{conv}\{f^{(1)}(t, x, u) : u \in \mathcal{U}\},$$

$$\mathcal{F}^{(2)}(t, x) = \text{conv}\{f^{(2)}(t, x, v) : v \in \mathcal{V}\}.$$

Let a certain convex closed set $\mathcal{X}$ be given in the phase space $\{x\}$, such that the boundary Γ of this set is a smooth surface, and $\mathcal{X} \setminus \Gamma$ is an open set. Suppose that the constraint $x[t] \in \mathcal{X}$ is imposed on the motion $x[t]$, which we shall treat as a kind of nonretaining holonomic constraint ⁽²⁾. A controlled system whose motion satisfies the constraint $x[t] \in \mathcal{X}$ will be called a **nonfree** controlled

system. To describe its motion, we proceed as follows. Let $n = n(x)$ be a vector-function defined by the conditions: if $x \in \Gamma$, then $n(x)$ is the unit vector of the normal to the surface Γ , inward with respect to $\mathcal{X}$, constructed at the point x ; if $x \notin \mathcal{X}$, then $n(x)$ is the inward unit normal to the surface Γ drawn through the point of the surface Γ nearest to x ; if $x \in \mathcal{X} \setminus \Gamma$, then $n(x) = 0$. Let

$$\mathcal{N}(t, x) = \begin{cases} \lambda n(x), \lambda \in [0, 2R(t, x)], & \text{if } x \in \mathcal{X}, \\ 2R(t, x)n(x), & \text{if } x \notin \mathcal{X}, \end{cases}$$

where $R(t, x)$ is the radius of the smallest Euclidean neighborhood of zero containing the algebraic sum of the sets $\mathcal{F}^{(1)}(t, x) + \mathcal{F}^{(2)}(t, x)$. Note that the function $\mathcal{N} = \mathcal{N}(t, x)$ is upper semicontinuous with respect to inclusion, and each of the sets $\mathcal{N}(t, x)$ is nonempty, convex, and closed.

Assuming that, when the point $x[t]$ moves along Γ , only normal reactions of the constraint arise, the motion of the nonfree controlled system (1), generated, for example, by an **approximating** strategy $U_a \div$

—by $\mathcal{U}_\Delta(t, x)$ ⁽¹⁾ and the program control of the second player $v = v(t) \in \mathcal{V}$, can be described by the contingent

$$dx_\Delta[t]/dt \in f^{(1)}(t, x_\Delta[t], u_\Delta[t]) + f^{(2)}(t, x_\Delta[t], v(t)) + \mathcal{N}(t, x_\Delta[t]),$$

$$u_\Delta[t] = u_\Delta[\tau_i] \in \mathcal{U}_\Delta(\tau_i, x_\Delta[\tau_i]) \quad \text{for } t \in [\tau_i, \tau_{i+1}).$$

The motion $x_\Delta[t]$ ($x_\Delta[t_0] = x_0$) of the nonfree controlled system, generated by the strategies U and V , will be denoted by the symbol $x_\Delta^{(n)}[t, t_0, x_0, U, V]$.

The results of paper ⁽¹⁾ carry over to the case of the game problem of bringing a nonfree controlled system to a given set $\mathcal{M} \subset \mathcal{X}$. This problem, in the class of approximal strategies, is formulated as follows. Let $U_a \div \mathcal{U}_\Delta(t, x)$ be some approximal strategy of the first player. We introduce for it the quality index

$$\gamma^0(U_a) = \sup_{\varepsilon > 0} \left| \limsup_{\delta \rightarrow 0} \left(\sup_{x_\Delta[t]} \vartheta_{x_\Delta[t]}^\varepsilon \right) \right|, \quad (3)$$

where $\vartheta_{x_\Delta[t]}^\varepsilon$ is the moment when, for the first time, $\rho(x_\Delta[t], \mathcal{M}) \leq \varepsilon$, $\rho(x, \mathcal{M})$ is the distance from x to $\mathcal{M}$, $\delta = \sup(\tau_{i+1} - \tau_i)$, $i = 0, 1, 2, \dots$, $x_\Delta[t] = x_\Delta^{(n)}[t, t_0, x_0, U_a, V_\tau]$, $V_\tau \div \mathcal{F}^{(2)}(t, x)$ is the trivial strategy of the second player ⁽¹⁾.

Problem 1. It is required to construct a minimax strategy $U_a^0 \div \mathcal{U}_\Delta^0(t, x)$, for which

$$\gamma^0(U_a^0) = \lim_{\varepsilon \rightarrow 0} \left(\inf_{\delta > 0} \left[\inf_{U_a} \left(\sup_{x_\Delta[t]} \vartheta_{x_\Delta[t]}^\varepsilon \right) \right] \right); \quad (4)$$

here U_a are all possible approximations of the first player, $x_\Delta[t] = x_\Delta^{(n)}[t, t_0, x_0, U_a, V_\tau]$.

We give some definitions which are analogues of the corresponding notions formulated in ⁽¹⁾ for free controlled systems.

Let a system of sets $\mathcal{W}(t)$ ($t_0 \leq t \leq \vartheta$) be given. We shall say that the system of sets $\mathcal{W}(t)$ is strongly u -stable for the nonfree system if $\mathcal{W}(t) \subset \mathcal{X}$ for $t_0 \leq t \leq \vartheta$ and, whatever $t_* \in [t_0, \vartheta]$, $w_* \in \mathcal{W}(t_*)$ and $\delta \in (0, \vartheta - t_*]$ may be, for any integrable function $v(t) \in \mathcal{V}$ there exists a motion $x^{(n)}[t, t_*, w_*, U_\tau, V_\Pi]$, for which $x^{(n)}[t_* + \delta, t_*, w_*, U_\tau, V_\Pi] \in \mathcal{W}(t_* + \delta)$; here $V_\Pi \div f^{(2)}(t, x, v(t))$ is a program strategy of the second player, $U_\tau \div \mathcal{F}^{(1)}(t, x)$ is the trivial strategy of the first player ⁽¹⁾.

In an analogous way the notion of u -stability ⁽¹⁾ of a system of sets $\mathcal{W}(t)$ is transferred. A strategy $U_a^{(e)} \div \mathcal{U}_\Delta^{(e)}(t, x)$, extremal to the system of sets $\mathcal{W}(t)$, is defined here in the same way as in paper ⁽¹⁾.

Lemma 1. *If $x_0 \in \mathcal{W}(t_0)$ and the system of sets $\mathcal{W}(t)$ ($t_0 \leq t \leq \vartheta$): (1⁰) is strongly u -stable for the nonfree system or (2⁰) is u -stable for the nonfree system and $\mathcal{W}(\vartheta) = \mathcal{M}$, then the strategy $U_a^{(e)} \div \mathcal{U}_\Delta^{(e)}(t, x)$ extremal to the system of sets $\mathcal{W}(t)$ ensures in case (1⁰) the condition*

$$\limsup_{\delta \rightarrow 0} \left[\sup_{x_\Delta[t]} \left(\max_{t_0 \leq t \leq \vartheta} \rho(x_\Delta[t], \mathcal{W}(t)) \right) \right] = 0, \quad (5)$$

and in case (2⁰)—the condition

$$\limsup_{\delta \rightarrow 0} \left[\sup_{x_\Delta[t]} \left(\min_{t_0 \leq t \leq \vartheta} \rho(x_\Delta[t], \mathcal{M}) \right) \right] = 0, \quad (6)$$

where $x_\Delta[t] = x_\Delta^{(n)}[t, t_0, x_0, U_a^{(e)}, V_\tau]$.

We shall say that, from the position $\{t_*, x_*\}$ ($t_0 \leq t_* \leq \vartheta$), the nonfree system **positionally absorbs the set $\mathcal{M}$ by the moment ϑ** , if

$$\sup_{\delta > 0} \left(\sup_{V_a} \left[\inf_{x_\Delta[t]} \left(\min_{t_0 \leq t \leq \vartheta} \rho(x_\Delta[t], \mathcal{M}) \right) \right] \right) = 0,$$

where $x_\Delta[t] = x_\Delta^{(n)}[t, t_*, x_*, U_\tau, V_a]$, and V_a are arbitrary approximation strategies of the second player.

By $\mathcal{W}^{(n)}(t, \vartheta)$ we denote the set of all points x for which the constrained system positionally absorbs $\mathcal{M}$ by the time ϑ from the position $\{t, x\}$.

Theorem 1. The collection of sets $\mathcal{W}^{(n)}(t, \vartheta)$ ($t_0 \leq t \leq \vartheta$) is u -stable for the constrained system. Let ϑ^0 be the smallest value of the parameter ϑ for which $x_0 \in \mathcal{W}^{(n)}(t_0, \vartheta)$. Then the strategy

$$U_a^{(e)} \div U_{\Delta}^{(e)}(t, x)$$

extremal to the system of sets $\mathcal{W}^{(n)}(t, \vartheta^0)$ ($t_0 \leq t \leq \vartheta^0$) is the desired minimax strategy of the first player.

In an analogous way, for the constrained system under consideration one can introduce the concepts of a maximin strategy of the second player

$$V_a^0 \div \mathcal{V}_{\Delta}^0(t, x),$$

the concepts of v -stability and strong v -stability of systems of sets, and formulate a theorem on the maximin—extremal strategy

$$V_a^0 \div \mathcal{V}_{\Delta}^{(e)}(t, x)$$

and on the saddle point of the game under consideration, analogous to the same theorem for the free controlled system (1).

Let us consider another type of game problem with a constraint on the phase vector $x[t]$. Suppose again that a set $\mathcal{M}$ is given, bringing the controlled system (1) to which is the aim of the first player. At the same time, the first player seeks to arrange the meeting of the point $x[t]$ with the set $\mathcal{M}$ so that the condition $x[t] \in \mathcal{H}$ is fulfilled, where $\mathcal{H}$ is some closed set containing $\mathcal{M}$. Thus, here the condition $x[t] \in \mathcal{H}$ is realized not by externally prescribed constraints, as was the case in problem 1, but must be ensured by a suitable choice of the control of the first player.

We shall call an approximation strategy of the first player $\mathcal{H}$ -admissible if the condition

$$\sup_{\varepsilon > 0} \left(\limsup_{\delta \rightarrow 0} \left[\sup_{x_{\Delta}[t]} \left(\max_{t_0 \leq t \leq \vartheta_{x_{\Delta}[t]}^{\varepsilon}} \rho(x_{\Delta}[t], \mathcal{H}) \right) \right] \right) = 0;$$

is satisfied. Here $\vartheta_{x_{\Delta}[t]}^{\varepsilon}$ is the instant of time when first $\rho(x_{\Delta}[t], M) \leq \varepsilon$, $x_{\Delta}[t] = x_{\Delta}[t, t_0, x_0, U_a, V_{\tau}]$, already being motions of the free system generated by the strategies U_a and V_{τ} .

For $\mathcal{H}$ -admissible strategies U_a we introduce the quality index $\gamma^0(U_a)$, defined by equality (3), where

$$x_{\Delta}[t] = x_{\Delta}[t, t_0, x_0, U_a, V_{\tau}].$$

We shall call an $\mathcal{H}$ -admissible strategy $U_a^0 \div \mathcal{U}_\Delta^0(t, x)$ minimax if equality (4) is satisfied, where on the right-hand side the inf is taken over all possible $\mathcal{H}$ -admissible approximation strategies U_a .

Problem 2. Among the $\mathcal{H}$ -admissible strategies

$$U_a \div \mathcal{U}_\Delta(t, x)$$

find a minimax strategy

$$U_a^0 \div \mathcal{U}_\Delta^0(t, x).$$

A system of sets $\mathcal{W}(t)$ ($t_0 \leq t \leq \vartheta$) will be called u -stable in $\mathcal{H}$ if $\mathcal{W}(t) \subset \mathcal{H}$ for $t \in [t_0, \vartheta]$ and, whatever $t_* \in [t_0, \vartheta]$, $w_* \in \mathcal{W}(t_*)$, and $\delta \in (0, \vartheta - t_*]$ may be, for any integrable function $\nu(t) \in \mathcal{P}$, among the motions

$$x[t, t_*, w_*, U_\tau, V_\Pi] \quad (\text{where } V_\Pi \div \mathcal{F}^{(2)}(t, x, \nu(t)))$$

there will be a motion $x(t)$ satisfying either the condition

$$x(t_* + \delta) \in \mathcal{W}(t_* + \delta),$$

or the condition

$$x(t) \in \mathcal{M} \quad \text{for } t \in [t_*, t_* + \delta].$$

In an analogous way one formulates the concept of strong u -stability in $\mathcal{H}$ (see the corresponding definition in (1)).

Lemma 2. If $x_0 \in \mathcal{W}(t_0)$, and the sets $\mathcal{W}(t)$, $t_0 \leq t \leq \vartheta$, are: (1°) strongly u -stable in $\mathcal{H}$ and $\mathcal{W}(\vartheta) = \mathcal{M}$, or (2°) u -stable in $\mathcal{H}$ and $\mathcal{W}(\vartheta) = \mathcal{M}$, then the strategy

$$U_a^{(e)} \div \mathcal{U}_\Delta^{(e)}(t, x)$$

extremal to the system of sets $\mathcal{W}(t)$ is $\mathcal{H}$ -admissible and, in case (1°), ensures condition (5), while in case (2°) it ensures equality (6), where $x_\Delta[t]$, $x_\Delta[t_0] = x_0$, are motions of system (1) generated by the strategies

$$U^{(e)} \div \mathcal{U}_\Delta^{(e)}(t, x) \quad \text{and} \quad V_\tau \div \mathcal{F}^{(2)}(t, x).$$

We shall say that, from the position $\{t_*, x_*\}$ ($t_0 \leq t_* \leq \vartheta$), the set $\mathcal{M}$ is **absorbed positionally** in $\mathcal{X}$ by the moment ϑ , if

$$\sup_{\delta > 0} \left(\sup_{V_a} \left[\inf_{x_\Delta[t], t_0 \leq t \leq \vartheta} (\min \rho(x_\Delta[t], \mathcal{M})) \right] \right) = 0;$$

here $x_\Delta[t] = x_\Delta[t, t_*, x_*, U_\tau, V_a]$ are motions satisfying the condition $x_\Delta[t] \in \mathcal{X}$ for all $t \in [t_*, t_{**}]$, where t_{**} is the moment when $x_\Delta[t] \in \mathcal{M}$ for the first time.

Let $\mathcal{W}(t, \vartheta \mid \mathcal{X})$ be the set of all points x for which the set $\mathcal{M}$ is positionally absorbed in $\mathcal{X}$ by the moment ϑ from the position $\{t, x\}$.

Theorem 2. The system of sets $\mathcal{W}(t, \vartheta \mid \mathcal{X})$ ($t_0 \leq t \leq \vartheta$) is u -stable in $\mathcal{X}$. Let ϑ^0 be the smallest value of the parameter ϑ for which $x_0 \in \mathcal{W}(t_0, \vartheta \mid \mathcal{X})$. Then the extremal strategy $U_a^{(e)} \doteq \mathcal{U}_\Delta^{(e)}(t, x)$ for the system of sets $\mathcal{W}(t, \vartheta^0 \mid \mathcal{X})$ is an $\mathcal{X}$ -admissible minimax strategy of the first player (i.e., it is a solution of problem 2).

The results of paper (1) can also be extended to the case of the problem of a maximum strategy V_a^0 , which must generate motions satisfying the phase constraint $x[t] \in \mathcal{X}$, and to the case of game problems with integral constraints on the resources of the players' control actions. For differential games of the type considered in the present paper, a classification described at the end of article (1) may be proposed.

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REFERENCES

1. N. N. Krasovskii, A. I. Subbotin, *DAN*, **190**, No. 3, 35 (1970).
2. F. R. Gantmakher, *Lectures on Analytical Mechanics*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.

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