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**Abstract**

**Full Text**

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*MATHEMATICS*

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## DIFFERENTIAL OPERATORS ON THE BASIC AFFINE SPACE

Let  $G$  be a connected semisimple algebraic Lie group of rank  $r$  over an algebraically closed field  $K$  of characteristic 0, let  $B$  be its Borel subgroup,  $N$  the unipotent radical of  $B$ , and  $H$  a Cartan subgroup contained in  $B$ .

A fundamental role in representation theory is played by the space  $A = N \backslash G$  — the basic affine space of the group  $G$ .  $A$  is an algebraic variety. The purpose of this paper is to study the space of differential operators with regular coefficients on  $A$ . For the precise definition of a regular differential operator see <sup>(1)</sup>.

To each function  $f(g)$  on the group  $G$  there will be constructed a set of regular differential operators on  $A$ . In addition, we shall show that in this way all differential operators on  $A$  can be obtained. For the construction an operation  $\pi_*$  is introduced, mapping functions on  $G$  to functions on  $A$ , which, in our opinion, is of independent interest. This operation is the algebraic analogue of the operation of averaging over a subgroup (in the present case, a unipotent one). Remarkably,  $\pi_*$  takes the operation of multiplication by a function  $f(g)$  into an “almost” differential operator on  $A$ . Precise formulations are given in Theorems 1 and 2.

1. Denote by  $\mathcal{E}(G)$  and  $\mathcal{E}(A)$  the spaces of regular functions on the algebraic varieties  $G$  and  $A$ . Define representations  $R^G$  and  $R^A$  of the group  $G$  in these spaces by right translations

$$(R_{g_0}^G f)(g) = f(gg_0), \quad (R_{g_0}^A \varphi)(x) = \varphi(xg_0), \quad g, g_0 \in G, x \in A.$$

Then every irreducible invariant subspace in these spaces is finite-dimensional. We can also introduce the representations  $L^G$  of the group  $G$  in  $\mathcal{E}(G)$  and  $L^A$  of the group  $H$  in  $\mathcal{E}(A)$  by left translations

$$(L_{g_0}^G f)(g) = f(g_0^{-1}g), \quad (L_h^A \varphi)(x) = \varphi(h^{-1}x), \quad g, g_0 \in G, h \in H, x \in A;$$

the element  $h^{-1}x$  is defined, since  $H$  normalizes  $N$ .

Let  $\mathfrak{G}$  be the Lie algebra of the group  $G$ ,  $\mathfrak{h}$  the Cartan subalgebra corresponding to  $H$ , and  $\mathfrak{N}$  the Lie algebra of the group  $N$ . Let  $\alpha_i$  ( $1 \leq i \leq k$ ) be all positive roots of  $\mathfrak{h}$  in  $\mathfrak{G}$ ,  $E_i^+ \in \mathfrak{N}$  the corresponding root vectors, and  $E_i$  the root vectors with weights  $\alpha_i$ .

Differentiating the representations  $L^G$  and  $L^A$ , we may regard elements of  $\mathfrak{G}$  as right-invariant differential operators of first order on  $G$ , and elements of  $\mathfrak{h}$  as differential operators of first order on  $A$ .

2. **The operation  $\pi_*$ .** Denote by  $\pi : G \rightarrow A = N \backslash G$  the natural projection and by  $\pi^* : \mathcal{E}(A) \rightarrow \mathcal{E}(G)$  the mapping given by the formula  $(\pi^*\varphi)(g) = \varphi(\pi g)$ . Then it is clear that  $R_g^G \pi^* = \pi^* R_g^A$  and  $L_h \pi^* = \pi^* L_h^A$ . Note, moreover, that  $\pi^*$  is a monomorphism and  $f \in \text{Im } \pi^*$  if and only if  $E_i f = 0$  for all  $i$ .

**Lemma-definition.** There exists a unique mapping  $\pi_* : \mathcal{E}(G) \rightarrow \mathcal{E}(A)$  such that

- 1)  $\pi_* R_g^G = R_g^A \pi_*$ ,  $\pi_* L_h^G = L_h^A \pi_*$  for all  $g \in G$ ,  $h \in H$ ;
- 2)  $\pi_* \pi^* \varphi = \varphi$  for all  $\varphi \in \mathcal{E}(A)$ .

**Proof.** We shall prove that  $\pi_* f$  is uniquely determined by conditions 1) and 2). Every regular function on  $G$  lies in a finite-dimensional irreducible subspace invariant with respect to  $L^G$ . Therefore it is enough to consider the case when  $f$  lies in an irreducible subspace  $V$  invariant with respect to  $L^G$  and is a weight function of weight  $\chi$  with respect to the restriction of  $L^G$  to  $H$ . If  $\chi$  is the highest weight of the given irreducible representation, then  $f \in \text{Im } \pi^*$ , i.e.  $f = \pi^* \varphi$ ,  $\varphi \in \mathcal{E}(A)$ , and, in view of 2),  $\pi_* f = \varphi$ .

Now let  $\chi$  not be the highest weight. Denote by  $f_0$  the vector of highest weight in  $V$  and by  $\chi_0$  this highest weight. Then  $f$  and  $f_0$  are transformed under the action of  $R^G$  according to one and the same irreducible representation of  $G$ . From 1) and from the fact that each irreducible representation of  $G$  occurs in  $\mathcal{E}(A)$  only once ((1), Lemma 4.1), it follows that  $\pi_* f$  and  $\pi_* f_0$  lie in one and the same irreducible subspace invariant with respect to  $R^G$ . But then the weights of  $\pi_* f$  and  $\pi_* f_0$  with respect to  $L^A$  coincide ((1), Lemma 4.2). From 1) it follows that the weight of  $\pi_* f$  is equal to  $\chi$ , while the weight of  $\pi_* f_0$  is equal to  $\chi_0 \neq \chi$ . Since  $\pi_* f_0 \neq 0$ , we have  $\pi_* f = 0$ .

This proof immediately also gives the construction of  $\pi_*$ . Namely, if  $f$  is a weight vector not of highest weight, lying in an irreducible subspace invariant with respect to  $L^G$ , then we set  $\pi_* f = 0$ . If, however,  $f$  is a vector of highest weight, then  $f = \pi^* \varphi$ , and we set  $\pi_* f = \varphi$ . The lemma is proved.

From the construction of  $\pi_*$  it is clear that  $\pi_*(\hat{E}_i f) = 0$  for any function  $f \in \mathcal{E}(G)$ .

**3. Construction with the aid of  $\pi_*$  of differential operators on  $A$ .** Let  $Wu$  be the enveloping algebra of the Lie algebra  $\mathfrak{h}$ . To each element  $w \in Wu$  let us associate right-invariant differential operators in the spaces  $\mathcal{E}(G)$  and  $\mathcal{E}(A)$ , which we shall denote by the same letter  $w$ . The equalities

$$w\pi^* = \pi^*w, \quad \pi_*w = w\pi_*$$

hold.

Let  $f \in \mathcal{E}(G)$ . Define in the space  $\mathcal{E}(A)$  the operator  $\bar{f}$  by the formula  $\bar{f}(\varphi) = \pi_*(f \cdot \pi^*\varphi)$ ,  $\varphi \in \mathcal{E}(A)$ .

**Theorem 1.** There exists a nonzero element  $w \in Wu$  such that  $w\bar{f}$  is a regular differential operator on  $A$ .

**Proof.** We shall call a chain a differential operator  $D$  on  $G$  of the form  $E_{i_1}E_{i_2}\dots E_{i_s}$ , and the weight of such a chain will be the weight  $a_{i_1} + \dots + a_{i_s}$ . Denote by  $\Xi$  the set of all chains  $D$  such that  $Df \neq 0$ , and by  $\Xi_0$  the set of their weights. Obviously,  $\Xi$  and  $\Xi_0$  are finite sets. For any function  $\varphi \in \mathcal{E}(A)$  and chain  $D$ ,

$D(f \cdot \pi^*\varphi) = Df \cdot \pi^*\varphi$ , since  $E_i\pi^*\varphi = 0$ . Therefore, if  $D \notin \Xi$ , then  $D(f\pi^*\varphi) = 0$ . Denote by  $U$  the subspace in  $\mathcal{E}(G)$  consisting of all functions  $u$  for which  $D(u) = 0$  for any chain  $D \notin \Xi$ .

**Lemma.** There exists a regular differential operator  $T$  on  $G$  and an element  $w \in Wu$  such that for all  $u \in U$  the equality

$$Tu = w\pi^*\pi_*u$$

holds.

The theorem follows immediately from the lemma, since for any function  $\varphi \in \mathcal{E}(A)$ ,  $f \cdot \pi^*\varphi \in U$ , and hence

$$T(f\pi^*\varphi) = w\pi^*\pi_*(f \cdot \pi^*\varphi) = \pi^*w\bar{f}(\varphi),$$

i.e.  $T \circ f \circ \pi^* = \pi^* \circ w \circ \bar{f}$ . From this equality it is clear that the differential operator  $T \circ f$  preserves  $\pi^*\mathcal{E}(A) \subset \mathcal{E}(G)$ , and therefore  $w \circ \bar{f}$  is a differential operator on  $A$ .

**Proof of the lemma.** Let  $H_1, \dots, H_r$  be a basis in  $\mathfrak{h}$ . The elements of  $Wu$  are polynomials in the  $H_i$ , and we shall regard them as

polynomial functions on  $\mathfrak{h}^*$ . Let  $\Delta$  be the second-order Laplace operator on  $G$  (constructed from the Killing form). Then there exists an element  $P \in Wu$  such that for any highest-weight vector  $\Psi \in \mathcal{E}(G)$  the equality  $\Delta\Psi = P\Psi$  holds, or, equivalently,  $\Delta\Psi = P(\chi_0)\Psi$ , where  $\chi_0$  is the weight of the vector  $\Psi$ .

Let  $B(H)$  be the restriction of the Killing form of the algebra  $\mathfrak{G}$  to  $\mathfrak{h}$ , and let  $Q(\chi)$  be the dual quadratic form on  $\mathfrak{h}^*$ . From the results of Harish-Chandra <sup>(2)</sup> it follows that  $P(\chi) = Q(\chi + \rho) - Q(\rho)$ , where  $\rho$  is the half-sum of the positive roots.

For any weight  $\beta$ , denote by  $P_\beta$  and  $w_\beta$  the elements of  $Wu$  corresponding to the polynomial functions

$$P_\beta(\chi) = P(\chi + \beta)$$

and

$$w_\beta(\chi) = 2\langle \beta, \chi + \rho \rangle + \langle \beta, \beta \rangle.$$

( $\langle, \rangle$ ) is the scalar product on  $\mathfrak{h}^*$  corresponding to the quadratic form  $Q$ .) Let

$$T = \prod_{\beta} (P_\beta - \Delta), \quad w = \prod_{\beta} w_\beta,$$

where  $\beta$  runs over  $\Xi_0 \setminus 0$ .

We shall show that for all  $u \in U$  the equality  $Tu = w\pi^*\pi_*u$  holds.

It is clear that it suffices to verify this equality when  $u$  is a weight vector lying in an irreducible  $L^G$ -invariant subspace  $V$ . Let  $u_0$  be a highest-weight vector in  $V$ , and let  $\chi$  and  $\chi_0$  be the weights of  $u$  and  $u_0$  relative to  $\mathfrak{h}$ . From the uniqueness of the highest-weight vector it follows that  $u_0 = cDu$ , where  $D$  is some chain,  $c \in K$ , and therefore  $\chi_0 - \chi \in \Xi_0$ .

1st case.  $\chi \neq \chi_0$ . The restriction of  $\Delta$  to  $V$  is multiplication by  $P(\chi_0)$ . Therefore

$$(P_\beta - \Delta)u = (P_\beta(\chi) - P(\chi_0))u = (P(\chi + \beta) - P(\chi_0))u = 0,$$

if  $\beta = \chi_0 - \chi \in \Xi_0 \setminus 0$ . Hence  $Tu = 0$ . Since in case 1  $\pi_*u = 0$ , also  $w\pi^*\pi_*u = 0$ , and hence  $Tu = w\pi^*\pi_*u$ .

2nd case.  $\chi = \chi_0$ . Then

$$\begin{aligned} (P_\beta - \Delta)u &= (P(\chi_0 + \beta) - P(\chi_0))u \\ &= (Q(\chi_0 + \beta + \rho) - Q(\chi_0 - \beta))u \\ &= (2\langle \chi_0 + \rho, \beta \rangle + \langle \beta, \beta \rangle)u = w_\beta(\chi_0)u. \end{aligned}$$

Thus,

$$Tu = \left( \prod_{\beta} (P_\beta - \Delta) \right) u = \prod_{\beta} w_\beta(\chi_0)u = wu = w\pi^*\pi_*u.$$

The proof of the lemma, and therefore also of Theorem 1, is complete.

It can be shown that the order of the operator  $w\bar{f}$  is equal to  $\nu(\Xi_0) - 1$ , i.e., to the order of  $w$ . ( $\nu(\Xi_0)$  is the number of elements of  $\Xi_0$ .)

**Theorem 2.** *Every regular differential operator  $\mathcal{D}$  on  $A$  can be represented in the form*

$$\mathcal{D} = \sum w_j \bar{f}_j, \quad \text{where } w_j \in Wu, \quad f_j \in \mathcal{E}(G).$$

**Proof.** As shown in <sup>(1)</sup>, §8, a regular differential operator  $\mathcal{D}$  on  $A$  extends to a regular differential operator  $\tilde{\mathcal{D}}$  on  $G$  (i.e.,  $\tilde{\mathcal{D}}\pi^*\varphi = \pi^*\mathcal{D}\varphi$  for all  $\varphi \in \mathcal{E}(A)$ ). The operator  $\tilde{\mathcal{D}}$ , like every differential operator on  $G$ , can be represented in the form

$$\tilde{\mathcal{D}} = \sum s_j(g)P_j,$$

where  $P_j$  are elements of the enveloping algebra  $\mathfrak{G}$ , and  $s_j \in \mathcal{E}(G)$ . By simple transformations  $\widetilde{\mathcal{D}}$  can be brought to the form

$$\widetilde{\mathcal{D}} = \sum w_j f_j + \sum A_i E_i + \sum \hat{E}_i B_i,$$

where  $A_i$  and  $B_i$  are some differential operators on  $G$ ,  $w_j \in Wu$ , and  $f_j \in \mathcal{E}(G)$ .

Since  $E_i \pi^* \varphi = 0$  and  $\pi_* \hat{E}_i f = 0$ , we have

$$\mathcal{D}\varphi = \pi_* \pi^* \mathcal{D}\varphi = \pi_* \widetilde{\mathcal{D}} \pi^* \varphi = \sum w_j \pi_*(f_j \pi^* \varphi) = \sum w_j \bar{f}_j(\varphi),$$

which proves Theorem 2.

Let  $L$  be the field of fractions of the ring  $Wu$ , and let  $R$  be the ring of regular differential operators on  $A$ . Then the operation  $f \mapsto \bar{f}$  extends to the mapping

$$\Theta : \mathcal{E}(G) \otimes_K L \rightarrow R \otimes_{Wu} L,$$

$\Theta$  commutes with the action of  $R_g$  and  $L_h$ .

**Theorem 3.**  $\Theta$  is an isomorphism of  $L$ -modules  $\mathcal{E}(G) \otimes_K L$  and  $R \otimes_{Wu} L$ .

**Proof.** As follows from (1), the dimension, over  $L$ , of the space of vectors transforming according to the given irreducible representation of the group  $G$  under the representations  $R^G$  and  $R^A$  in both spaces coincides. By Theorem 2,  $\Theta$  is an epimorphism; hence  $\Theta$  is an isomorphism.

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