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Abstract

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MATHEMATICS

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ON THE UNIQUENESS OF THE SOLUTION OF THE TRICOMI PROBLEM FOR ONE CLASS OF NONLINEAR EQUATIONS

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Let D be a simply connected bounded domain in the plane of the variables x, y with boundary $\Gamma = \sigma + AC + CB$, where σ is an open Jordan curve in the half-plane $y > 0$ with endpoints at $A(0, 0)$, $B(1, 0)$. The curves AC and CB lie in the lower half-plane and are given by the equations

$$x - \frac{2}{m+2}(-y)^{(m+2)/2} = 0, \quad x + \frac{2}{m+2}(-y)^{(m+2)/2} = 1$$

respectively. The number m is odd and positive.

In this domain we consider the equation

$$T[u] \equiv y^m u_{xx} + u_{yy} = f(x, y, u), \quad (1)$$

where $f(x, y, u)$ is a function which, for all $(x, y) \in \overline{D}$ and for all $|u| < \infty$, satisfies the following conditions: $f(x, y, u)$ and $f_u(x, y, u)$ are continuous,

$$f_u(x, y, u) > 0, \quad (2)$$

$$f_u(x, y, u) \leq m(m+4)4^{-(m+4)/(m+2)}(m+2)^{-2m/(m+2)}y^{-2} \quad \text{for } y < 0. \quad (3)$$

Equation (1) is elliptic for $y > 0$, degenerates parabolically for $y = 0$, is hyperbolic for $y < 0$, and the curves AC and CB are its characteristics.

By $\varphi(s)$ we denote a continuous function prescribed on σ of the arc length s , measured from the point $B(1, 0)$ in the positive direction. The function $\psi(x, y)$

is defined for $(x, y) \in \overline{D}$, $y \leq 0$ and is continuously differentiable once. In addition, we introduce the notation $D_1 = D \cap \{y > 0\}$; $D_2 = D \cap \{y < 0\}$. By D^ε we denote the subdomain of D obtained by cutting off from D a piece by the curves σ_ε and $B_\varepsilon C_\varepsilon$, where

$$\sigma_\varepsilon : (x-1)^2 + \frac{4}{(m+2)^2} y^{m+2} = \varepsilon^2, \quad B_\varepsilon C_\varepsilon : x + \frac{2}{m+2} (-y)^{(m+2)/2} = 1 - \varepsilon.$$

Accordingly, we shall denote

$$D_1^\varepsilon = D^\varepsilon \cap \{y > 0\}, \quad D_2^\varepsilon = D^\varepsilon \cap \{y < 0\}.$$

Tricomi problem. It is required to find a solution $u(x, y)$ of equation (1), regular in D and continuous in $\overline{D}$, satisfying the conditions

$$u|_\sigma = \varphi, \quad (4)$$

$$u|_{AC} = \psi|_{AC}, \quad (5)$$

and moreover its first derivatives may have integrable singularities at the point A and on the closed arc BC .

Theorem 1. *If conditions (2), (3) are satisfied, then problem (1), (4), (5) can have only one solution.*

The validity of this assertion follows directly from the following theorem.

Theorem 2. *If conditions (2), (3) are satisfied, then any solution of problem (1), (4), (5), if it exists, satisfies the estimate*

$$|u| \leq kK, \quad (6)$$

where $k > 0$ is a constant depending on the diameter of the domain D , and the number K is determined

is determined as follows:

$$K = \max_{\overline{D}} |f(x, y, 0)| + \max_{\overline{\sigma}} |\varphi(s)| + \max_{AC} |\psi(x, y)| + \max_{AC} |x^{m/(m+2)} \psi_\eta|, \quad (7)$$

where

$$\psi_\eta = -\frac{1}{2} \psi_x + \frac{1}{2} (-y)^{-m/2} \psi_y.$$

First consider the equation

$$T[u] = -c(x, y)u + F(x, y) \quad (8)$$

and prove the validity of estimate (6) for it.

Theorem 3. *If the function $z(x, y)$ satisfies the conditions*

$$z(x, y) \in C(\overline{D}) \cap C^1(D \cup AC) \cap C^2(D),$$

$$T[z] + C(x, y)z = E(x, y) \geq 0, \quad (9)$$

$$z_\eta(x, y)|_{AC} \geq 0, \quad (10)$$

then the greatest positive value of the function $z(x, y)$, if it exists, is attained on $\overline{\sigma}$.

To prove this assertion we shall use the following known theorems.

Theorem (see ^(1, 2)). If conditions (2), (3), (9), (10) are satisfied, then the function $z(x, y)$ can attain its greatest positive value in the closed domain $\overline{D}^\varepsilon$ only on the segment $y = 0$, $0 \leq x \leq 1 - \varepsilon$.

Theorem (see ⁽²⁾). Under the conditions of the preceding theorem, if $z(x, y)$ attains its greatest positive value in $\overline{D}_2$ at the point $(x_0, 0)$, $0 < x_0 < 1$, then

$$z_y(x_0, 0) > 0. \quad (11)$$

Proof of Theorem 3. Fix an arbitrary number $\varepsilon > 0$. According to the condition of the theorem, $z(x, y) \in C(\overline{D})$, and therefore it is continuous in the closed subdomain $\overline{D}^\varepsilon$. Denote its greatest positive value in $\overline{D}^\varepsilon$ by M_ε . According to the first theorem, we have

$$0 \leq \max_{\overline{D}^\varepsilon} z(x, y) = \max_{AB_\varepsilon} z(x, y) = z(x_0, 0), \quad 0 \leq x_0 \leq 1 - \varepsilon,$$

where, naturally, $z(x_0, 0) \leq M_\varepsilon$. By the known maximum principle for elliptic equations (see ⁽³⁾) we have

$$\max_{\overline{D}^\varepsilon} z(x, y) = \max_{\overline{\sigma}_0 \cup AB_\varepsilon \cup \overline{\sigma}_\varepsilon} z(x, y) \leq M_\varepsilon,$$

where

$$\sigma_0 = \sigma \setminus \left\{ (x-1)^2 + \frac{4}{(m+2)^2} y^{m+2} \leq \varepsilon^2 \right\}.$$

Thus, we obtain

$$\sup \left\{ \max_{\overline{D}^\varepsilon_2} z(x, y), \max_{\overline{D}^\varepsilon_1} z(x, y) \right\} = \max_{\overline{D}^\varepsilon} z(x, y) = \max_{\overline{\sigma}_0 \cup \overline{AB}_\varepsilon \cup \overline{\sigma}_\varepsilon} z(x, y). \quad (12)$$

On the other hand, $z(x, y)$ is continuous in the closed domain $\overline{D}_1$, twice continuously differentiable in D_1 , and satisfies condition (9). According to the maximum principle,

$$\max_{\overline{D}_1} z(x, y) = \max_{\overline{\sigma} \cup \overline{AB}} z(x, y),$$

and, consequently,

$$\max_{\overline{\sigma}_\varepsilon} z(x, y) \leq \max_{\overline{\sigma} \cup \overline{AB}} z(x, y). \quad (13)$$

From (12) and (13) we obtain

$$\max_{\overline{D}^\varepsilon} z(x, y) \leq \max_{\overline{\sigma} \cup \overline{AB}} z(x, y) = \max_{\overline{D}_1} z(x, y).$$

Suppose that $\max_{\overline{D}_1} z(x, y)$ is attained on AB at some point $(x_0, 0)$, $0 < x_0 < 1$. Then, according to Zaremba's lemma ^(2,4), at this point we shall have $z_y(x_0, 0) < 0$, which contradicts inequality (11). Consequently, our supposition is false. Thus, finally we obtain

$$M_\varepsilon = \max_{\overline{\sigma}} z(x, y)$$

for arbitrary $\varepsilon > 0$, as was required to prove.

Proof of Theorem 2 for equation (8). Consider the function

$$h(y) = K(\alpha - \beta e^{2y}), \quad (14)$$

where

$$\beta > \sup \left\{ (1 + e^{2\overline{y}}), \left[1 + 2 \left(\frac{2}{m+2} \right)^{m/(m+2)} e^{-1} \right] \right\}, \quad \overline{y} = \max_{\overline{D}} |y|, \quad \alpha > 1 + \beta e^{2\overline{y}}.$$

By direct calculation we obtain

$$h_{\eta}(y) = -\beta(-y)^{-m/2}e^{2y}K,$$

$$-h_{\eta}(y)|_{AC} = \beta \left(\frac{2}{m+2} \right)^{m/(m+2)} \exp \left[-2 \left(\frac{m+2}{2} x \right)^{2/(m+2)} \right] x^{-m/(m+2)} K.$$

Hence, multiplying both sides by $x^{m/(m+2)}$, we have

$$-x^{m/(m+2)}[h_{\eta}(y)]_{AC} \geq K \geq \max_{AC} |x^{m/(m+2)}\psi_{\eta}|,$$

i.e.

$$-x^{m/(m+2)}[h_{\eta}(y)]_{AC} \geq |x^{m/(m+2)}\psi_{\eta}|_{AC},$$

whence it follows that

$$-x^{m/(m+2)}[h_{\eta}(y)]_{AC} \geq -x^{m/(m+2)}[\psi_{\eta}(x, y)]_{AC}, \quad (15)$$

$$-x^{m/(m+2)}[h_{\eta}(y)]_{AC} \geq x^{m/(m+2)}[\psi_{\eta}(x, y)]_{AC}. \quad (16)$$

Further,

$$-\{T[h] + c(x, y)h\} = \{4\beta e^{2y} - c(\alpha - \beta e^{2y})\}K \geq K \geq \max_D |F(x, y)|. \quad (17)$$

In addition, the inequalities

$$h|_{\sigma} \geq \max_{\sigma} |\varphi(s)|, \quad (18)$$

$$h|_{AC} \geq \max_{AC} |\psi(x, y)|. \quad (19)$$

are obvious.

Take the function $z(x, y) = u(x, y) - h(y)$, where $u(x, y)$ is a solution of problem (4), (5), (8). For it, according to (17) and (15), we have

$$T[z] + cz = F(x, y) - \{T[h] + ch\} \geq F(x, y) + \max_D |F(x, y)| \geq 0, \quad z_{\eta}(x, y)|_{AC} \geq 0.$$

From (18) and (19), respectively, the inequalities follow

$$z(x, y)|_{\sigma} \leq 0, \quad z(x, y)|_{AC} \leq 0. \quad (20)$$

Thus, the function $z(x, y)$ satisfies all the requirements of Theorem 3. Therefore it must be that

$$0 \leq \max_{\overline{D}} z(x, y) = \max_{\sigma} z(x, y),$$

whence, on the basis of inequality (20), we obtain

$$z(x, y) = u(x, y) - h(y) \leq 0 \quad \text{everywhere in } \overline{D}. \quad (21)$$

Next consider $z_1(x, y) = -u(x, y) - h(y)$, for which, analogously, we obtain $T[z_1] + cz_1 \geq -F(x, y) + \max |F(x, y)| \geq 0$. On the basis of condition (16) we have $(z_1)_{\eta}|_{AC} \geq 0$, and from (18), (19) it follows that

$$z_1(x, y)|_{\sigma} \leq 0, \quad z_1(x, y)|_{AC} \leq 0. \quad (22)$$

According to Theorem 3 we may write

$$z_1(x, y) = -u(x, y) - h(y) \leq 0 \quad \text{everywhere in } \overline{D}. \quad (23)$$

Inequalities (22) and (23) give $u(x, y) \leq h(y)$, $-h(y) \leq u(x, y)$, whence we finally have $|u(x, y)| \leq h(y)$. Denoting $k = \max_{\overline{D}}(a - \beta e^{2y})$, we obtain estimate (6).

It is now easy to prove Theorem 2. Let there exist a solution of the Tricomi problem (1), (4), (5), namely $u(x, y)$. Substituting it into equation (1), we obtain the identity $T[u] \equiv f(x, y, u)$, which can be written in the form (5) $T[u] - f_u(x, y, \tilde{u}) = f(x, y, 0)$, where $\tilde{u}$ lies between zero and $u(x, y)$. In the identity thus obtained, the coefficient of $u(x, y)$ satisfies conditions (2), (3). Referring to the assertion proved above, we obtain the estimate $|u| \leq h$, which completes the proof of Theorem 2.

Suppose that the Tricomi problem (1), (4), (5) has two solutions $u_1(x, y)$ and $u_2(x, y)$. Then the difference $v(x, y) = u_1(x, y) - u_2(x, y)$ vanishes on the curves σ and AC , respectively. Consequently, $v_{\eta}|_{AC} = 0$. Moreover, we have the identity (see (6))

$$\begin{aligned} T[v] &\equiv T[u_1] - T[u_2] \equiv f(x, y, u_1) - \\ &- f(x, y, u_2) = (u_1 - u_2) \int_0^1 \Phi_{u_1}(x, y, u_1, t) \frac{dt}{t}, \quad \text{where } \Phi(x, y, u_1, t) = \\ &= f(x, y, u_2 + t(u_1 - u_2)). \end{aligned}$$

The coefficient of $v = u_1 - u_2$ satisfies the required conditions.

On the basis of Theorem 2 it follows that $v(x, y) = 0$ everywhere in $\overline{D}$, i.e. $u_1(x, y) \equiv u_2(x, y)$. The uniqueness theorem is proved.

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