

NONLINEAR EQUATIONS OF HAMMERSTEIN TYPE WITH A MONOTONE OPERATOR

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Abstract

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MATHEMATICS

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NONLINEAR EQUATIONS OF HAMMERSTEIN TYPE WITH A MONOTONE OPERATOR

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1. In the present paper we study a nonlinear equation of the form

$$x + BFx = f \quad (1)$$

in a complex Hilbert space H . Here the nonlinear operator F is assumed to be bounded, i.e. to transform every bounded set into a bounded set, and monotone. The linear operator B has a domain of definition $D(B)$ dense in H and is closed. Equations of this kind, under the assumption of self-adjointness of the operator B , were studied in ⁽¹⁻⁵⁾, with the operator B assumed bounded in ⁽¹⁻³⁾ and unbounded in ^(4,5). The main difference of the present paper from those listed above is that here the operator B is assumed to be neither self-adjoint nor bounded, while the monotonicity requirement on the operator F is weakened. In the first part of this paper new existence and uniqueness theorems for equation (1) are established, and in the second part, under uniqueness conditions, strong convergence to the solution of approximations of Galerkin-Petrov type is proved (see, for example, ^(6,7)). We note that the starting point for the first part was provided by the papers ^(4,5,8). The proof uses a theorem of Browder ⁽⁹⁾, Minty ⁽¹⁰⁾.

2. **Theorem 1.** *Suppose the following conditions are fulfilled:*

1°. B is a linear closed operator acting in the Hilbert space H , with dense domain of definition $D(B)$, and

$$\operatorname{Re}(Bx, x) \geq 0 \quad \text{for } \forall x \in D(B),$$

$$\operatorname{Re}(B^*y, y) \geq 0 \quad \text{for } \forall y \in D(B^*).$$

2°. F is a nonlinear hemicontinuous bounded operator.

3°. There exists a strongly continuous operator G such that

$$\operatorname{Re}(Fx - Fy, x - y) \geq \operatorname{Re}(Gx - Gy, x - y) \quad \text{for } \forall x, y \in H,$$

4°. F is coercive, i.e.

$$\operatorname{Re}(F_f x, x) / \|x\| \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty.$$

$$F_f x = F(x + f).$$

Then equation (1) has at least one solution for any $f \in H$. This solution is unique when either

$$\operatorname{Re}(Fx - Fy, x - y) \geq 0, \quad \operatorname{Re}(Bx, z) > 0 \quad \text{for } 0 \neq x \in D(B),$$

or

$$\operatorname{Re}(Fx - Fy, x - y) > 0 \quad \text{for } x \neq y.$$

From this theorem there follow, in particular, the following propositions.

Theorem 2. If B is a self-adjoint nonnegative operator, and F satisfies the conditions of Theorem 1, then equation (1) has a solution.

This proposition differs little from Theorem 5 of the paper ⁽⁵⁾.

Theorem 3. If B is a bounded operator, $\operatorname{Re}(Bx, x) \geq 0$, and F satisfies the conditions of Theorem 1, then equation (1) has a solution.

Theorem 3 is a certain modification of Theorem 4.7 from the work ⁽⁸⁾.

3. Let the Hilbert space H be the orthogonal sum of two of its subspaces H_+ and H_- , and let the operator B , with dense domain of definition, satisfy the conditions

$$B(D(B) \cap H_+) \subset H_+, \quad B(D(B) \cap H_-) \subset H_-,$$

where the restrictions of the operator B to H_+ and H_- , denoted by B_+ and $-B_-$, have the properties:

$$\begin{aligned} \operatorname{Re}(B_+ x, x) &\geq 0 & \text{for } \forall x \in D(B), \\ \operatorname{Re}(B_+^* y, y) &\geq 0 & \text{for } \forall y \in D(B^*), \\ \operatorname{Re}(B_- x, x) &\geq m \|x\|^2 & \text{for } \forall x \in D(B) \cap H_-, \\ \operatorname{Re}(B_-^* y, y) &\geq m \|y\|^2 & \text{for } \forall y \in D(B^*) \cap H_-, \end{aligned}$$

where m is some nonnegative number.

Theorem 4. Let the operator B be closed and satisfy the conditions listed above. Then, if F is a hemicontinuous bounded and strongly monotone operator:

$$\operatorname{Re}(Fx - Fy, x - y) \geq a\|x - y\|^2 \quad \text{for } \forall x, y \in H,$$

where $am > 1$, equation (1) has a unique solution for any $f \in H$.

In the case when B is a self-adjoint or normal operator, the space H decomposes naturally into an orthogonal sum of subspaces corresponding to the parts of the spectrum lying to the right and to the left of the imaginary axis. If, moreover, the negative spectrum is separated from the imaginary axis ($\operatorname{Re} \lambda < -m < 0$), then the operator B is representable as a sum of operators satisfying the conditions of Theorem 4.

We therefore arrive at the propositions:

Theorem 5. If B is a self-adjoint operator whose spectrum is located outside the interval $(-m, 0)$, where m is some positive number, and F satisfies the conditions of Theorem 4, then equation 1 has a unique solution (cf. (5)).

Theorem 6. If B is a normal operator whose spectrum is located outside the strip $-m < \operatorname{Re} \lambda < 0$ for some $m > 0$, and F satisfies the conditions of Theorem 4, then equation (1) has a unique solution.

4. Here we shall describe a Galerkin-Petrov type method for the approximate determination of a solution of equation (1). The application of such a method is associated with a difficulty connected with the choice of the initial coordinate system, each element of which must, as is known, belong to the domain of definition of the operator appearing on the left-hand side of the equation. In our case the domain of definition of the operator $(I + BF)$ is not known in advance. If the operator F is invertible, which will be the case, for example, under strong monotonicity, then the domain of definition $D(I + BF) = F^{-1}(D(B))$. In view of this we shall construct a certain modification of the Galerkin-Petrov method. We shall assume that each element of the complete linearly independent system $\{\varphi_k\}_1^\infty$ belongs to the domain of definition of the operator B , and, moreover, if Q_n is the orthoprojector onto the subspace H_n spanned by the first n vectors, then

$$\|Q_n B Q_n x - Bx\| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \forall x \in D(B). \quad (2)$$

We shall seek an approximate solution in the form $x_n = \sum_{k=1}^n a_k^n \varphi_k$.

Theorem 7. Let the operator B be as in Theorem 1, and let the hemicontinuous bounded operator F satisfy the condition

$$\operatorname{Re}(Fx - Fy, x - y) \geq \gamma(\|x - y\|)\|x - y\|,$$

where $\gamma(t)$ is a nonnegative, nondecreasing function for $t > 0$, tending to infinity as $t \rightarrow \infty$. Let the system $\{\varphi_k\}_1^\infty$ be as described above. Then each equation

$$x + Q_n B Q_n F x = Q_n f \quad (n = 1, 2, 3, \dots) \quad (3)$$

has a unique solution $x_n \in H_n$, and the sequence $\{x_n\}$ converges strongly to the solution x_0 of equation (1):

$$\|x_n - x_0\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4)$$

In the case where the operator B is such that the form $\operatorname{Re}(Bx, x)$ does not preserve sign, additional conditions must be imposed on the coordinate system. Namely, if B satisfies the conditions of Theorem 4, then we shall assume the following:

- a) Each element φ_k belongs either to H_+ or to H_- , and in the latter case $\varphi_k \in D(B) \cap D(B^*)$ (i.e., in this case we assume in advance that, for the operator B , the set $W = d(B) \cap D(B^*)$ is dense in H_-).
- b) For every $x \in D(B)$,

$$\|Q_n B Q_n x - Bx\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5)$$

- c) For every $y \in D(B_-^*)$,

$$\|Q_n B_-^* Q_n y - B_-^* y\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (6)$$

Theorem 8. Let the operators B and F satisfy the conditions of Theorem 4 and let $D(B) \cap D(B^*) \supset H_-$. Then, if the coordinate system of linearly independent vectors $\{\varphi_k\}_1^\infty$ is complete in H and satisfies conditions a), b), c), then for each n equation (3) has a unique solution belonging to the linear span of the first n vectors of the system, and

$$\|x_n - x_0\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (7)$$

Let us note that the rate of convergence of the approximations in (4) and (7) depends on the properties of the coordinate system—on the rate of convergence to zero in (2), (5), and (6).

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REFERENCES

1. M. M. Vainberg, *Uspekhi Mat. Nauk*, **10**, issue 4 (66), 189 (1955).
2. M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, 1956.
3. M. M. Vainberg, *Siberian Mathematical Journal*, **2**, No. 2, 201 (1961).
4. I. M. Lavrent'ev, *Dokl. Akad. Nauk SSSR*, **166**, No. 2, 284 (1966).
5. I. M. Lavrent'ev, *Dokl. Akad. Nauk SSSR*, **175**, No. 6, 1219 (1967).
6. G. I. Petrov, *Prikl. Mat. Mekh.*, **4**, 3, 3 (1940).
7. G. I. Petrov, *Prikl. Mat. Mekh.*, **21**, 184 (1957).
8. C. L. Dolph, G. J. Minty, Nonlinear Integral Equations, *Proc. of an Advanced Seminar Conducted by Mathematics Research Center*, Madison, 1964, p. 99.
9. F. E. Browder, *Bull. Am. Math. Soc.*, **69**, No. 6, 862 (1963).
10. G. J. Minty, *Proc. Nat. Acad. Sci. U.S.A.*, **50**, No. 6, 1038 (1963).

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