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Abstract

Full Text

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MATHEMATICS

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ON $L(1, \chi)$ WITH A REAL DIRICHLET CHARACTER ON SPARSE SETS OF VALUES OF THE CHARACTER MODULUS

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The study of the functions

$$L(1, \chi_D) = \sum_{n=1}^{\infty} \chi(D, n) n^{-1},$$

where $\chi(D, n)$ denotes: $\chi_n(D) = \left(\frac{\pm D}{n}\right)$, the Jacobi symbol or zero; or $\chi_D(n) = \left(\frac{\pm D}{n}\right)$, the Kronecker symbol or zero, was carried out in ⁽¹⁻⁸⁾ and is connected with applications to the theory of quadratic forms and the theory of divisors of quadratic fields. Recently the problem of the distribution of values of $L(1, \chi_D)$ has acquired additional interest in connection with A. Selberg's formula ⁽⁹⁾ for the trace of Hecke operators in the space of modular forms.

In the present paper new information is given on the nature of the behavior of the functions $L(1, \chi_D)$, obtained on the basis of the method indicated in ⁽¹⁰⁾ for estimating double sums with a quadratic Dirichlet character.

Notation: $D = dm + l$, $1 \leq l \leq d$, $(l, d) = 1$, $m = 1, 2, \dots$; b_D is any sequence of zeros and ones; $\gamma = 2$ or 1 , according as $\chi(D, n) = \chi_n(D)$ or $\chi_D(n)$; $\tau_k(m)$ is the number of solutions of the equation $m = m_1 \dots m_k$ in integers $m_\nu \geq 1$,

$$H_k(N) = \sum_{m=1}^N \left[b_D \prod_{p|d} \left(1 - \frac{\chi(D, p)}{p} \right) L(1, \chi_D) \right]^k,$$

where p is a prime number.

Main lemma. Uniformly with respect to N, d , and l we have

$$H_k(N) = \sum_{\substack{n=1 \\ (n, \gamma)=1}}^N \sum_{\substack{m=1 \\ (D, n)=1}}^N \frac{\tau_k(n^2) \chi(d^2, n) b_D}{n^2} + O(N^{9/10} d^{4/5} \ln^c dN) +$$

$$+O\left(\ln^c dN \left[\sum_{2 \leq n \leq X} \sum_{\substack{r=1 \\ (r,n)=1}}^n \left| \sum_{\substack{m=1 \\ D \equiv r \pmod{n}}}^N b_D - F \right|^2 \right]^{1/2}\right),$$

where F is an arbitrary quantity not depending on l ; $c = c(k)$; $X = N^{2/3}d^{3/10}$.

Thus the problem of the asymptotics for the moments of the function

$$b_D \prod_{p|d} \left(1 - \frac{\chi(D, p)}{p}\right) L(1, \chi_D) \quad (1)$$

reduces only to the equidistribution of the sequence b_D “on average” over primitive arithmetic progressions. As is known, a broad class of sequences possesses a property of this kind. In particular, for $b_D = \mu^2(D)$, where μ is the Möbius function, the following is obtained.

Theorem 1. Uniformly with respect to N , d , and l we have

$$H_k(N) = N \sum_{\substack{n=1 \\ (n,\gamma)=1}}^{\infty} \sum_{r/n} \sum_{\delta=1}^{\infty} \frac{\tau_k(n^2) \chi(d', n) \mu(r) \mu(\delta) \varphi(r)}{(nr\delta)^2} + O\left(N^{8/9} d^{4/5} \ln^C dN\right),$$

where $\varphi(r)$ is Euler’s function.

With the aid of this theorem one can obtain rather varied information concerning the distribution of the values of the function $L(1, \chi_D)$. For example, from it and from the theorem of Fréchet and Shohat ^(11,12) there follows a limit theorem for the distribution of the functions (1) in arithmetic progressions whose difference $d \leq N^{1/9}$ grows together with N .

As consequences of Theorem 1 one can also obtain information of a different, less usual, nature. Namely, let henceforth d denote the product of the primes not exceeding $q \geq 2$. In this case the following holds.

Theorem 2. For an arbitrary integer $k \geq 1$, uniformly with respect to N , $d \leq N^{1/9}$, $1 \leq l \leq d$, $(l, d) = 1$, we have

$$\sum_{m=1}^N \mu^2(D) L^k(1, \chi_D) = N \prod_{p|d} \left(1 - \frac{\chi(l, p)}{p}\right)^{-k} \left[1 + O\left(\frac{\ln^2 q}{q}\right)\right].$$

This theorem, in particular, shows that the quantity $\mu^2(D) L(1, \chi_D)$ on the progressions $dm + l$ is, on the average, independent of m and is determined only by the difference of the progression d and its initial term l . It is very plausible that

in fact the quantity $L(1, \chi_D)$ on the indicated set of values D is, in the main, always equal to

$$\prod_{p/d} \left(1 - \frac{\chi(l, p)}{p}\right)^{-1}.$$

In any case, the following is certainly true.

Theorem 3. Under the conditions of Theorem 2, for each $m = 1, 2, \dots, N$ with $\mu^2(dm + l) = 1$, except for at most $N/\ln^{1-\varepsilon} N$ of them, where $\varepsilon > 0$,

$$L(1, \chi_{dm+l}) = \prod_{p/d} \left(1 - \frac{\chi(l, p)}{p}\right)^{-1} \left[1 + O\left(\frac{1}{\ln \ln 2d}\right)\right].$$

Hence, with the aid of the known Mertens formulas, one obtains asymptotic expressions for the “large” and “small” values of $L(1, \chi_D)$. Namely, there exist $l = l'$ and $l = l''$ such that, for almost all values D in the sense of Theorem 3, we have

$$L(1, \chi_D) = e^c \ln \ln D + O(1)$$

and, respectively,

$$L(1, \chi_D) = \frac{\pi^2}{6e^c \ln \ln D} \left(1 + O\left(\frac{1}{\ln \ln D}\right)\right),$$

where c is Euler’s constant.

These relations constitute a substantial strengthening of the known inequalities ^(1–8) for $L(1, \chi_D)$, both in the sense of estimating the approximation to the main term of the growth of $L(1, \chi_D)$, and in the sense of the density of the values D for which they hold.

If in the basic lemma one takes $b_D = 1$ when $dm + l$ is a prime number, and $b_D = 0$ otherwise, then from this lemma and from recent results of the “large sieve” in the form of Theorem 8 of ⁽¹³⁾ there will follow entirely analogous results concerning the values of $L(1, \chi_D)$ on the set of primes $D = dm + l$.

Applications to the theory of quadratic forms and divisors of quadratic fields consist in a simple combination of the estimates given here with the corresponding exact Dirichlet formulas (14) for the number of classes of non-equivalent quadratic forms of discriminant $\pm D$ and the number of divisor classes of quadratic fields.

Let, for example, h and R denote, respectively, the class number and the regulator of the quadratic field of discriminant D . Then, for almost all values of D in the sense of Theorem 3, the following asymptotic law holds:

$$\ln hR = \ln \sqrt{|D|} - \ln \prod_{p|d} \left(1 - \frac{\chi(l, p)}{p}\right) - \frac{1 + \operatorname{sign} E}{2} \ln 2 - \frac{1 - \operatorname{sign} D}{2} \ln \pi + O\left(\frac{1}{\ln \ln 2d}\right).$$

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REFERENCES

1. J. Littlewood, Proc. Lond. Math. Soc., 2 (1928).
 2. R. Paley, J. Lond. Math. Soc., 3 (1938).
 3. Yu. V. Linnik, DAN, 37, No. 4 (1942).
 4. A. Walfish, Trans. Inst. Math. Tbilisi, No. 11 (1942).
 5. S. Chowla, Proc. Nat. Acad. Sci. India, 13 (1947).
 6. S. Chowla, Proc. Lond. Math. Soc. (2), 50 (1949).
 7. P. T. Bateman, S. Chowla, P. Erdős, Publ. Math., 1, Debrecen (1949–1950).
 8. M. B. Barban, UMN, 21, issue 1 (127) (1966).
 9. A. Selberg, Sborn. per. Matematika, 1, 4 (1957).
 10. A. F. Lavrik, DAN, 186, No. 1 (1969).
 11. M. Frechet, J. Shohat, Trans. Am. Math. Soc. 33 (1931).
 12. M. B. Barban, Izv. AN SSSR, ser. matem., 26, 4 (1962).
 13. K. Prachar, *Distribution of Prime Numbers*, Moscow, 1967.
 14. Z. Borevich, I. R. Shafarevich, *Lectures on Number Theory*, Moscow, 1966.
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