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M. I. Rabinovich

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Abstract

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MATHEMATICAL PHYSICS

M. I. RABINOVICH

ON AN ASYMPTOTIC METHOD IN THE THEORY OF NONLINEAR OSCILLATIONS OF DISTRIBUTED SYSTEMS

(Presented by Academician A. V. Gaponov-Grekhov on 26 IX 1969)

In the present paper an asymptotic method is considered for studying oscillatory and wave processes in weakly nonlinear distributed systems, analogous to Bogolyubov's asymptotic method for concentrated systems ⁽¹⁾. Problems concerning only one-dimensional systems are discussed, although the general approach and some results can also be extended to spatially non-one-dimensional systems.

As is known, Bogolyubov's method can be applied directly to the study of bounded weakly nonlinear distributed systems ⁽²⁻⁴⁾. For this purpose a single-frequency solution is represented as a sum of normal oscillations, whose parameters—slow functions of time—are described by a system of ordinary differential equations. There exists, however, a broad class of problems for which the indicated approach is not adequate and its application is difficult. These are problems associated with the propagation and interaction of waves in weakly nonlinear systems with dispersion (including bounded ones, if the wavelengths are much smaller than the length of the system).

In the proposed method, the parameters of an approximate solution of the nonlinear problem are regarded as slow functions not only of time but also of the coordinate, and approximate equations in partial derivatives are obtained for them. It should be noted that, for a number of concrete problems, various versions of shortened partial differential equations have already been used ⁽³⁻¹¹⁾. In the present paper a method, related to the averaging method, is proposed for obtaining these equations in the general case.*

Usually the equations describing fields in a weakly nonlinear medium (or the motion of a continuous medium) can be represented in the form

$$a_k \frac{\partial u_k}{\partial t} + \sum_{l=1}^n \left(a_{kl} \frac{\partial}{\partial x} + b_{kl} \right) u_l = \mu f_k^{(1)}(u, u'_x, u'_t, \tau, \chi, t, x) + \mu^2 f_k^{(2)} + \dots,$$

$$k = 1, 2, \dots, n, \quad \tau = \mu t, \quad \chi = \mu x, \quad \mu \ll 1, \quad (1)$$

where the functions $f_k^{(i)}$ are polynomials in u , u'_x , u'_t and rapidly oscillating periodic functions of the explicitly entering x and t .

For $\mu = 0$ system (1) has solutions in the form of traveling waves

$$u_k \sim \psi_k^m e^{i(\omega t - kx)} + \text{c.c.}, \quad (2)$$

where ψ_k^m are determined from the system

$$\sum_{l=1}^n (a_k \omega \delta_{kl} - a_{kl} k_m - ib_{kl}) \psi_l^m = 0, \quad (3)$$

* A similar approach in connection with the averaging method was first proposed by A. V. Gaponov and the author in 1968.

and the relation between the complex frequency ω and the wave number $k = 2\pi/\lambda$ (λ is the wavelength) is determined by the dispersion equation

$$\text{Det} \|a_k \omega \delta_{kl} - a_{kl} k_m - ib_{kl}\| \equiv D(\omega, k) = 0; \quad (4)$$

here m is the index of the normal wave (branch of the dispersion equation). It is assumed that for $\mu = 0$ propagating waves exist in the system, i.e., if ω, k are complex, then $\text{Im } \omega / \text{Re } \omega \sim \text{Im } k / \text{Re } k \sim \mu$.

For $\mu \neq 0$ the multiwave solution—the result of the interaction of quasiharmonic waves (wave packets with spectral width $\Delta\omega/\omega \sim \Delta k/k \sim \mu$)—is sought in the form

$$u_k = \sum_{s=1}^q \left\{ \sum_{m=1}^r \psi_k^m(\omega_s) A_s^m(\tau, \chi) \exp \{i [\omega_s t - k_m(\omega_s) x + \varphi_s^m(\tau, \chi)]\} + \text{complex conjugate} \right\} + \mu w_k^{(1)}(\tau, \chi, x, t) + \dots + \mu^n w_k^{(n)}(\tau, \chi, x, t) + \dots, \quad (5)$$

where ω and k satisfy (4), and $w_k^{(n)}$ are periodic functions of the explicitly entering variables x, t . In the solution (5) one must take into account those waves arising because of the nonlinearity which are in resonance (with respect to spatial and temporal periods) with the normal waves of the linear system.

We seek equations for the unknown functions $A(\tau, \chi)$ and $\varphi(\tau, \chi)$ in the form

$$\partial A_s^m / \partial t = \mu F_1^{sm} \{A, \varphi, \tau, \chi\} + \mu^2 F_2^{sm} \{A, \varphi, \tau, \chi\} + \dots,$$

$$\partial \varphi_s^m / \partial t = \mu \Phi_1^{sm} \{A, \varphi, \tau, \chi\} + \mu^2 \Phi_2^{sm} \{A, \varphi, \tau, \chi\} + \dots \quad (6)$$

Here F and Φ are unknown differential operators in partial derivatives. The solution (5) can be determined with any prescribed degree of accuracy if the unknown periodic functions $w_k^{(n)}(x, t)$ are found and the form of the operators F_n, Φ_n is revealed. We shall find the equations for $w_k^{(n)}$ from (1), substituting (5) and (6) into it and equating the coefficients of like powers of μ :

$$a_k \frac{\partial w_k^{(1)}}{\partial t} + \sum_{l=1}^n \left(a_{kl} \frac{\partial}{\partial x} + b_{kl} \right) w_l^{(1)} = h_k^{(1)}(\tau, \chi, x, t),$$

$$h_k^{(1)} = - \sum_{s=1}^q \exp(i\omega_s t) \left\{ \sum_{m=1}^r \exp[-i(k_{mx} - \varphi_s^m)] [(a_k \psi_k^m F_1^{sm} + \sum_{l=1}^n a_{kl} \psi_l^m \frac{\partial A_s^m}{\partial \chi}) + i A_s^m \left(a_k \psi_k^m \Phi_1^{sm} + \sum_{l=1}^n a_{kl} \psi_l^m \Phi_l^{sm} \right) + \text{c.c.} \right\} + f_k^{(1)}(u^0, u_x^0, u_t^0, \tau, \chi, x, t); \quad (7)$$

.....

$$a_k \frac{\partial w_k^{(n)}}{\partial t} + \sum_{l=1}^n \left(a_{kl} \frac{\partial}{\partial x} + b_{kl} \right) w_l^{(n)} = h_k^{(n)}(\tau, \chi, x, t),$$

$$h_k^{(n)} = -a_k \frac{\partial w_k^{(n-1)}}{\partial \tau} - \sum_{l=1}^n a_{kl} \frac{\partial w_l^{(n-1)}}{\partial \chi} + \sum_{l=1}^n \frac{1}{n!} f_{ku_l}^{(1)n} w_l^{(n)} + \dots$$

$$\dots + \sum_{l=1}^n f_{ku_l}^{(n-1)'} w_l^{(1)} - a_k \sum_{s=1}^q \exp(i\omega_s t) \sum_{m=1}^r \exp\{-i(k_{mx} - \varphi_s^m)\} \times \psi_k^m (F_n^{sm} + i A_s^m \Phi_n^{sm}) + f_k^{(n)}(u^0, u_x^0, u_t^0, \tau, \chi, x, t)$$

where u^0 is determined from (5) for $w^{(i)} = 0$. Next we shall make use of the periodicity of the right-hand sides of these equations in x and t and represent them in the form

$$h_k(\tau, \chi, x, t) = \sum_{s=1}^q \left(\sum_{m=1}^r H_k^{sm}(\tau, \chi, \omega, k) \exp(-ik_m x) \right) \exp(i\omega_s t) + \sum_{d=1}^{d_0} \left(\sum_{l=1}^{l_0} H_k^{dl}(\tau, \chi, \omega, k) \exp(-ik_l x) \right) \exp(i\omega_d t)$$

In the first group of terms here we have separated the summands corresponding to the natural waves of the system; for them

$$D(\omega_s, k_m) = 0, \quad (10)$$

and in the second group—the waves arising because of nonlinearity, which are not in resonance with the first ones; for them

$$D(\omega_d, k_l) \neq 0. \quad (11)$$

The functions H_k^{ab} are defined as Fourier coefficients $\left(ab^{sm}_{dl}\right)$

$$H_k^{ab} = \frac{1}{T\Lambda} \int_t^{t+T} \int_x^{x+\Lambda} h_k(\tau, \chi, xt) \exp[-i(\omega_a t - k_b x)] dx dt. \quad (12)$$

The periodic functions $w_k(x, t)$ are also represented in the form (9) ($h_k \rightarrow w_k$, $H_k \rightarrow W_k$). After substituting the expressions for w_k and h_k into (7), (8), equating the coefficients of exponentials with identical exponents, we obtain an inhomogeneous system of algebraic equations for determining $W_k^{(n)}$. The amplitudes $W_k^{(1)dl}$, in accordance with (11), are bounded and equal to

$$W_k^{(1)dl} = \sum_{j=1}^n D_{jk} H_j^{(1)dl} / D(\omega_d, k_l)$$

(D_{jk} is the minor of D). Bounded solutions for $W_k^{(1)sm}$, however, in view of (10), can exist, as is known, only when the orthogonality conditions are satisfied

$$\sum_{l=1}^n \zeta_l^m(\omega_s) H_l^{(1)sm} = 0, \quad s = 1, 2, \dots, q, \quad m = 1, 2, \dots, r, \quad (13)$$

where ζ_l^m are the eigenfunctions of the system adjoint to (3). From these conditions, taking into account (7), (9), we find the required functionals F_1^{sm} and Φ_1^{sm} and obtain the equations of the first approximation

$$\begin{aligned} \partial A_s^m / \partial t + v_{gr}^m(\omega_s) \partial A_s^m / \partial x &= \mu \operatorname{Re} f_{sm}^{(1)}, \\ \partial \varphi_s^m / \partial t + v_{gr}^m(\omega_s) \partial \varphi_s^m / \partial x &= (\mu / A_s^m) \operatorname{Im} f_{sm}^{(1)}, \end{aligned} \quad (14)$$

where

$$f_{sm}^{(i)} = (\gamma_m / D_p) \sum_{l=1}^n \zeta_l^m f_l^{(i)sm}, \quad \gamma_m = \text{const},$$

$$f_l^{(i)sm} = \frac{1}{T\Lambda} \int_t^{t+T} \int_x^{x+\Lambda} f_l^{(i)}(u^0, u_x^0, u_t^0, \tau, \chi, x, t) \exp[-i(\omega_s t - k_m x + \varphi_s^m)] dx dt,$$

and

$$v_{gr} = D'_k / D'_p$$

is the quantity usually called the group velocity ($p = i\omega$, $\chi = -ik$), while in a dispersive system $v_{gr} = d\omega/dk \neq v_\varphi = \omega/k$. We note that equations (14) were obtained in the case when, for $\mu = 0$, ω_s and k_m are real. If, however, ω_s (or k_m) is complex, $\omega_s = \omega'_s + i\sigma_s$, then, introducing a new variable $A_s^m = A_{s\text{cmar}}^m e^{-\sigma_s t}$, we obtain for it an equation differing from (14) only in that an additional term $A_s^m \sigma_s$ is present in its right-hand side.

To construct the equations of the second approximation with respect to A_s^m and φ_s^m , it is necessary to determine the functions $W_k^{(1)sm}$. Using (7), (14), we find

$$W_k^{(1)sm} = e^{i\varphi_s^m} [\psi'_{kx} - v_{gr}^m(\omega_s) \psi'_{kp}] \left(\frac{\partial A_s^m}{\partial x} + iA_s^m \frac{\partial \varphi_s^m}{\partial x} \right) + \frac{1}{D'_p} \sum_{l=1}^n \left[D'_{lkp} f_l^{(1)sm} - \frac{\partial}{\partial p} (a_l D_{lk} \psi_l) f_{sm}^{(1)} \right].$$

From the boundedness condition for $W_k^{(2)}$ (see (13)), F_2^{sm} and Φ_{2s}^m are determined, which makes it possible to find the equations of the second approximation. They turn out to be parabolic and are written in the form*

$$\frac{\partial a_s^m}{\partial t} + v_{gr}^m(\omega_s) \frac{\partial a_s^m}{\partial x} - \frac{i}{2} \frac{d^2 \omega}{dk^2} \bigg|_{\omega_s^m k_m} \frac{\partial^2 a_s^m}{\partial x^2} = \mu f_{sm}^{(1)} + \mu^2 \left\{ f_{sm}^{(2)} + [t_u^{(1)} w^{(1)}]_{sm} + [f'_{u_x}{}^{(1)}(u_x^0 + w_x^{(1)})]_{sm} + [f'_{u_t}{}^{(1)}(u_t^0 + w_t^{(1)})]_{sm} \right\}$$

Here $a_s^m = A^m \exp(i\varphi_s^m)$ is the complex amplitude of the wave;

$$[f'_{u_x}{}^{(1)}(u_x^0 + w_x^{(1)})]_{sm} = \frac{\gamma_m}{T\Lambda D'_p} \int_t^{t+T} \int_x^{x+\Lambda} \sum_{k=1}^n \xi_k^m \sum_{l=1}^n f'_{u_x}{}^{(1)} \left(\frac{\partial w_l^{(1)}}{\partial x} + \frac{\partial u_l^0}{\partial x} \right) \times \exp[-i(\omega_s t - k_m x + \varphi_s^m)] dx dt,$$

$[f_{u_t}^{(1)'} w^{(1)}]_{sm}$ and $[f'_{u_t}{}^{(1)}(u_t^0 + w_t^{(1)})]_{sm}$ are determined analogously. In (15), in addition to v_{gr} , there enters one more parameter characterizing the dispersion in the system, $d^2\omega/dk^2 = \beta$. If in the given spectral region $\beta = 0$, then the equations of the second approximation also turn out to be hyperbolic, differing from (14) by terms of the type $f(a)\partial a/\partial x$, $f(a)\partial a/\partial t$. Another case very important

for applications is when $\beta \neq 0$ and $f_k^{(1)} = 0$. It corresponds to consideration of broad wave packets $-\Delta\omega/\omega \sim \Delta k/k \sim \mu$, while $f_k \sim f_k^{(2)} \sim \mu^2$. In this case, (15) differ in form from the equations of the first approximation only by the presence of the term with the second derivative.

Equations of higher approximations are constructed analogously to (15). The question of convergence of the proposed asymptotic method remains open for the time being.

Scientific-Research Radiophysics Institute
at Gorky State University

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* In deriving (15) it was taken into account that

$$\rho = \gamma_m \sum_{l=1}^n \xi_l \left[a_l (2\psi'_{lp} v_{gr}^m - \psi'_{lx}) - \sum_{q=1}^n a_{lq} \psi'_{qp} \right], \quad \frac{\partial}{\partial p} f_l^{sm} = 0.$$

Note: Figure translations are in progress. See original paper for figures.

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