

ON BICOMPACTNESS AND COMPLETENESS OF TOPOLOGICAL SPACES IN CONNECTION WITH THE THEORY OF ABSOLUTES

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Abstract

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MATHEMATICS

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ON BICOMPACTNESS AND COMPLETENESS OF TOPOLOGICAL SPACES IN CONNECTION WITH THE THEORY OF ABSOLUTES

(Presented by Academician P. S. Aleksandrov, 18 V 1970)

1. Let us call an **H -system** any family ξ of nonempty χa -sets*, directed by inclusion, i.e., satisfying the condition: for any two elements $A' \in \xi$, $A'' \in \xi$ there is an element $A \in \xi$ such that $A \subset A' \cap A''$. Maximal H -systems, as usual, are called **H -ends**.

The paper gives a new criterion for bicomcompactness.

Theorem 1. *For the bicomcompactness of a regular space it is sufficient (and, obviously, necessary) that every H -system have a nonempty intersection.*

In this theorem, which generalizes the elementary theorem on nested segments, the class of centered systems of sets for which a nonempty intersection is required has apparently been reduced to a minimum.

Remark 1. From the maximality property of H -ends it follows that if a χa -set A' contains some element $A \in \xi$, then $A' \in \xi$ itself.

Let X be an arbitrary space. Denote by $H(X)$ the set of all H -ends of the space X . On this set we introduce the classical topology of P. S. Aleksandrov (¹). For an arbitrary χa -set A of the space X , denote by O_A the family of all H -ends containing the set A as an element. The collection of the sets O_A , where A ranges over all χa -sets of the space X , will be taken as an open base of the space $H(X)$. It is easy to verify that in this way a topology is indeed defined.

We shall prove that the space $H(X)$ for any space X is a nonempty bicomcompactum. This follows from the following theorem.

Theorem 2. *The limit of the complete refinement** $s_\chi X$ of the maximal finite spectrum $s_\chi X$ of an arbitrary space X is the space $H(X)$.*

Proof. First of all, from the very definition of an H -system it follows that no H -system can contain two different elements belonging to one and the same partition α .

Proposition 1. *The set of all elements of a zero-dimensional thread $\xi = \{A_\alpha\}$ of the spectrum $s_\chi X$ is an H -end $\eta = \eta(\xi)$. Conversely, if $\eta = \{A^\gamma\}$ is some H -end, then for any $\alpha \in \chi_X$ the intersection $\eta \cap \alpha$ consists of a single element A_α , and $\xi = \{A_\alpha\}$ is a zero-dimensional thread for which $\eta(\xi) = \eta$.*

Proof of Proposition 1. If $\xi = \{A_\alpha\}$ is a zero-dimensional thread of the spectrum $s_\chi X$, then, taking arbitrarily $A_\alpha \in \xi$, $A_{\alpha'} \in \xi$, and α'' , following both α and α' , we obtain $A_{\alpha''} \in \xi$, contained both in A_α and

* χa -sets are sets that are closures of open sets (canonical closed sets).

** In the sense of V. I. Ponomarev ^(2,3): the complete refinement of a projection spectrum $s = \{K_\alpha, \tilde{\omega}_\alpha^{\alpha'}\}$ is the spectrum $s' = \{K'_\alpha, \tilde{\omega}_\alpha^{\alpha'}\}$, obtained from s by replacing each complex K_α by the zero-dimensional complex K'_α , consisting of all vertices of the complex K_α . The maximal finite spectrum is the spectrum $s_\chi X = \{N_\alpha, \tilde{\omega}_\alpha^{\alpha'}\}$, consisting of the nerves N_α of all finite partitions α of the space X . Here a partition (according to Ponomarev) of the space X is called any locally finite covering consisting of χa -sets with disjoint open cores. The directed set of all, respectively all finite, partitions of the space X is denoted by ξ_X , respectively χ_X .

and in $A_{\alpha'}$. Thus, the set of elements of the zero-dimensional thread ξ is an H -system $\eta(\xi)$. The fact that $\eta(\xi)$ is a maximal H -system follows from the following lemma.

Main lemma. If η is an H -end, then for every $\alpha \in \chi_x$ the intersection $\eta \cap \alpha$ is nonempty (and hence consists of a single element).

Indeed, suppose this lemma has been proved and suppose $\eta = \eta(\xi)$ is contained in an H -system η' distinct from it, which we may assume to be maximal. Let $A' \in \eta'$, $A' \notin \eta$, and let $\alpha \in \chi_x$ be some partition containing the element A' . Since (by the lemma) α also contains some element $A \in \eta$ and $A' \neq A$, it follows that α contains at least two elements of the H -system η' , which is impossible.

For the proof of the main lemma we need

Lemma 1. If $\eta = \{A_\lambda\}$ is an H -end and A is such an x -set that

$$IA \cap IA_\lambda \neq \Lambda$$

for every $A_\lambda \in \eta$, then $A \in \eta$.

The assertion follows from the fact that, adjoining to the H -system η all elements of the form $[IA \cap IA_\lambda]$, we again obtain an H -system.

From Lemma 1 it follows that

Lemma 2. If A' and A'' are elements of the H -end η , then also $[IA' \cap IA''] \in \eta$.

We shall first prove the main lemma for the case of a partition $\alpha = \{A, B\}$ consisting of two elements (then obviously $B = [X \setminus A]$).

Take an arbitrary $A_\lambda \in \eta$ and consider the case when $IA_\lambda \cap IA = \Lambda$. Then also $IA_\lambda \cap A = \Lambda$, i.e. $IA_\lambda \subseteq B$, $A_\lambda \subseteq B$, hence $B \in \eta$. It remains to consider the case when $IA_\lambda \cap IA \neq \Lambda$ for every $A_\lambda \in \eta$. But then by Lemma 1 we have $A \in \eta$.

Assume that the formula $\eta \cap \alpha \neq \Lambda$ has been proved for all partitions α consisting of m elements, and let us prove it for an arbitrary partition

$$\alpha = \{A_1^\alpha, \dots, A_m^\alpha, A_{m+1}^\alpha\}, \quad m \geq 2, \quad (1)$$

consisting of $m+1$ elements. Keeping the notation of formula (1), consider the partition

$$\alpha' = \{A_1^{\alpha'}, \dots, A_{m-1}^{\alpha'}, A_m^{\alpha'}\},$$

where $A_i^{\alpha'} = A_i^\alpha$ for $i \leq m-1$ and $A_m^{\alpha'} = A_m^\alpha \cup A_{m+1}^\alpha$.

By the induction hypothesis, one of the elements of the partition α' belongs to η . If this element is one of the sets $A_i^{\alpha'}$, $i \leq m-1$, then everything is proved.

Let $A_m^{\alpha'} \in \eta$. Consider the partition $\alpha'' = \{A_1^{\alpha''}, A_2^{\alpha''}\}$, $A_1^{\alpha''} = A_m^{\alpha'}$,

$$A_2^{\alpha''} = X \setminus IA_1^{\alpha''} = [X \setminus A_1^{\alpha''}].$$

If $A_1^{\alpha''} \in \eta$, then again everything is proved.

Let $A_2^{\alpha''} \in \eta$. We shall show that $[IA_2^{\alpha''} \cap IA_m^{\alpha'}] = A_m^\alpha$. Indeed,

$$A_m^{\alpha'} = A_m^\alpha \cup A_{m+1}^\alpha, \quad A_2^{\alpha''} = X \setminus IA_{m+1}^\alpha = \bigcup_{k=1}^m A_k^\alpha.$$

Furthermore,

$$IA_m^{\alpha'} = I(A_m^\alpha \cup A_{m+1}^\alpha) = X \setminus \bigcup_{j=1}^{m-1} A_j^\alpha, \quad IA_2^{\alpha''} = X \setminus A_{m+1}^\alpha,$$

$$\begin{aligned} [IA_m^{\alpha'} \cap IA_2^{\alpha''}] &= \left[\left(X \setminus \bigcup_{j=1}^{m-1} A_j^\alpha \right) \cap (X \setminus A_{m+1}^\alpha) \right] \\ &= [X \setminus (A_1^\alpha \cup \dots \cup A_{m-1}^\alpha \cup A_{m+1}^\alpha)] = [IA_m^\alpha] = A_m^\alpha, \end{aligned}$$

as required. By Lemma 2 we have $A_m^\alpha \in \eta$; the main lemma is proved.

The second assertion of Proposition 1 is now proved as follows.

Let an H -end η be given. For each $\alpha \in \mathcal{N}_x$ put $\eta \cap \alpha = (A_\alpha)$. It is required to prove that for $\alpha'' > \alpha'$ we have $A^{\alpha''} \subseteq A^{\alpha'}$. Since η is an H -end, there exists $A \in \eta$, $A \subseteq A^{\alpha'} \cap A^{\alpha''}$. Since $\alpha'' > \alpha'$ and $A^{\alpha''} \in \alpha''$, there is a unique element of the partition α' containing the set $A^{\alpha''}$, and since $A \subseteq A^{\alpha'} \cap A^{\alpha''}$, this element can only be $A^{\alpha'}$.

Thus, we have an (obviously one-to-one) mapping

$$\eta: \tilde{s}_\chi X \rightarrow H(X)$$

of the space $\tilde{s}_\chi X$ onto $H(X)$.

It is easy to verify that, in this case,

$$\eta(O_c) = O_A,$$

where e is some vertex of the spectrum $s_\chi X$ corresponding to the χa -set A . From this equality we conclude that the mapping η carries an open base of the space $\tilde{s}_\chi X$ onto an open base of the space $H(X)$. Consequently, η is a homeomorphism. Theorem 2 is proved.

Let us call an H -end $\xi = \{A_\lambda\}$ an h -end if $\bigcap A_\lambda \neq \Lambda$.

The set of all h -ends of the space X forms a subspace $h(X)$ of the space $H(X)$. Consequently, $h(X)$ is completely regular.

A result of V. Ponomarev is known ^(2,3), according to which, for a regular space X , the space $\tilde{s}_\chi X$ coincides with the Čech extension βX^* of the absolute X^* , while the absolute X^* itself is the subspace of the space $\tilde{s}_\chi X$ consisting of all zero-dimensional threads whose elements have nonempty intersection. These threads correspond to H -ends whose elements have nonempty intersection, i.e., to h -ends.

Theorem 3. *The absolute X^* of a regular space X is the space $h(X)$; the space $H(X)$ is the Čech extension βX^* of the absolute X^* .*

If every H -system in a regular space X has nonempty intersection, then $X^* = h(X) = H(X)$, i.e., the absolute of the regular space X is bicomact; consequently, X itself is bicomact. We have proved Theorem 1*.

2. Denote by $s_\zeta X = \{N_\alpha \mathfrak{D}_\alpha^{\alpha'}\}$ the spectrum over the family ζ_X of all partitions of the space X ^(2,3). Since $\chi_X \subseteq \zeta_X$, the spectrum $s_\chi X$ is a subspectrum of the spectrum $s_\zeta X$.

Definition. Let us call a centered system of χa -sets ξ a ζ -system if it satisfies the following conditions:

- A) if $A \in \xi$ and σ is some set of χa -sets forming a partition of the set A , then $\xi \cap \sigma \neq \Lambda$;
- B) if $A \in \xi$ and A lies in a χa -set A' , then also $A' \in \xi$.

A space is called ζ -complete if in it the intersection of every ζ -system is nonempty.

Proposition 2. *Every paracompactum is a ζ -complete space.*

Proof. Following V. I. Ponomarev ⁽⁴⁾, say that a system of sets ξ touches a cover α if the cover α contains an element intersecting all the sets belonging to

the system ξ . Ponomarev observed that, for a space X to be paracompact, it is necessary and sufficient that every system of closed sets touching all locally finite covers have nonempty intersection. He also proved that, in this condition, locally finite covers may be replaced by partitions.

Now let ξ be an arbitrary ζ -system. Then in every $\alpha \in \zeta_X$ there is an element $A \in \xi$, and this element intersects all the elements of the system ξ . In other words, every ζ -system touches all partitions $\alpha \in \zeta_X$. If X is paracompact, then by Ponomarev's criterion it follows that the intersection of all $A \in \xi$ is nonempty, i.e., that X is ζ -complete.

Definition. Threads of the spectrum $s_\zeta X$ (respectively of the spectrum $s_\chi X$) will be called ζ -threads (respectively χ -threads). A zero-dimensional thread $\xi = \{A^\alpha\}$ is called nonempty if all elements forming the thread have nonempty intersection.

* One can also give a direct proof of this theorem; however, in essence it repeats the lemmas on which the proof of the basic properties of the absolute rests.

Proposition 3. In a regular space X , every nonempty zero-dimensional $\varkappa$ -thread $\xi_\varkappa$ is uniquely completed to a zero-dimensional ζ -thread $\xi_\zeta = \zeta(\xi_\varkappa)$; moreover, the threads $\xi_\zeta = \zeta(\xi_\varkappa)$ and $\xi_\varkappa$ consist of one and the same set of elements.

Proof. We first prove that in every partition $\alpha \in \zeta_\varkappa$ there is an element A^α (obviously unique) belonging to the given nonempty thread $\xi_\varkappa$. Let x_0 be the unique point belonging to all elements $A \in \xi_\varkappa$, and let $\sigma = \{A_1^\alpha, \dots, A_s^\alpha\}$ be the totality of all elements of the partition α that contain the point x_0 . Put $\sigma' = \alpha \setminus \sigma$, $B = \sigma' = [X \setminus \bar{\sigma}]$. Then among the elements of the finite partition $\{A_1^\alpha, \dots, A_s^\alpha, B\}$ of the space X , exactly one is contained in $\xi_\varkappa$. This element cannot be B , since $x_0 \notin B$. Hence one of the elements $A_1^\alpha, \dots, A_s^\alpha$ of the cover α is contained in $\xi_\varkappa$. Choosing, for each α , the element $A^\alpha \in \xi_\varkappa$, we obtain a ζ -thread. Indeed, the set of all elements $\xi_\varkappa$ is an H -system by Proposition 1. Therefore for $\alpha', \alpha'', \alpha'' > \alpha'$, there is an $A \in \xi_\varkappa$ lying in $A^{\alpha'} \cap A^{\alpha''}$. But $\alpha'' > \alpha'$, and $A^{\alpha''}$ is contained in a unique element of the partition α' and has no interior points in common with any other element of this partition. Therefore from $A \subset A^{\alpha'} \cap A^{\alpha''}$ it follows that $A^{\alpha''} \subset A^{\alpha'}$, i.e., for $\alpha'' > \alpha'$ we have $\omega_{\alpha'}^{\alpha''} A^{\alpha''} = A^{\alpha'}$. The proposition is proved.

On the other hand, it is obvious that if X is a regular ζ -complete space, then every zero-dimensional ζ -thread ξ_ζ contains a unique nonempty $\varkappa$ -thread $\xi_\varkappa$, and $\zeta(\xi_\varkappa) = \xi_\zeta$.

Thus, for ζ -complete spaces X there exists a one-to-one correspondence between zero-dimensional nonempty $\varkappa$ -threads and all zero-dimensional ζ -threads. By virtue of this correspondence we have a natural homeomorphism between the spaces $h(X)$ and $\tilde{s}_\zeta X$; thereby proved is

Theorem 4. The absolute of a regular ζ -complete space X coincides with the limit of the complete relaxation of the spectrum $s_\zeta X$.

Remark 2. Theorem 4 is also valid for a (at least formally) broader class of so-called ζ^φ -complete spaces*; these are spaces in which nonemptiness of intersection is required only of ζ^φ -systems, i.e., ζ -systems ξ for which all $\xi \cap \alpha$, $\alpha \in \zeta_\omega$, are finite.

For paracompact spaces, Theorem 4, as is known, was proved by V. I. Ponomarev (^{2,3}), not directly, but via the descriptive definition of the absolute as the maximal perfect preimage of the space X .

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* I do not know whether there exist ζ^φ -complete spaces that are not ζ -complete.

Note: Figure translations are in progress. See original paper for figures.

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