

ON SOME PROPERTIES OF $\setminus(R\setminus)$ -, $\setminus(R^c\setminus)$ -OPERATIONS AND PROJECTIVE OPERATIONS ON UNCOUNTABLE SYSTEMS OF SETS IN CONNECTION WITH BRANCHING HYPOTHESES

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Abstract

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MATHEMATICS

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ON SOME PROPERTIES OF R -, R^c -OPERATIONS AND PROJECTIVE OPERATIONS ON UNCOUNTABLE SYSTEMS OF SETS IN CONNECTION WITH BRANCHING HYPOTHESES

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In the paper ⁽¹⁾ we considered certain properties of R - and R^c -operations on countable systems of sets. Analogous properties of certain operations were considered in ⁽²⁾. In the present note analogous questions are considered for the same operations on uncountable systems of sets. We use the notation introduced in note ⁽¹⁾.

1. Let $T = \langle \mathcal{E}, < \rangle$ be a branching table, $U \subset \mathcal{E}$, $T^* = \langle U(T), < \rangle$. If $\rho(T^*) = \omega_\lambda$, ω_τ is the final character of the number ω_λ , and $F = (x_\beta)_{\beta < \omega_\lambda}$ is a monotone dissection of the table T' , then $(\forall \beta < \omega_\lambda) \cdot \overline{[(x_\beta, \cdot)_{T^*} \cap U]} \geq \aleph_\tau$, and further either $\overline{F} \cap \overline{U} = \aleph_\tau$, or there exists a disjunctive subset $K \subset U$ such that $(\forall \beta < \omega_\lambda) \cdot \overline{[(x_\beta, \cdot)_{T^*} \cap K]} \geq \aleph_\tau$.

In what follows, by $\aleph_\nu$ we shall denote a strongly inaccessible cardinal number. If $\overline{E} = p$, then E will be called a p -set. The following hypotheses are used in the paper:

(α_1) Every branching table T of rank ω_ν such that $(\forall \alpha < \omega_\nu) \cdot \overline{[T_\alpha]} < \aleph_\nu$ attains its rank.

(α_2) If in a branching table $T = \langle \mathcal{E}, < \rangle$ of rank ω_ν the cardinality of every disjunctive subset $U \subset \mathcal{E}$ is less than $\aleph_\nu$, then the table T attains its rank ⁽³⁾.

These hypotheses are respectively equivalent to the following:

(β_1) Every branching table T of rank ω_ν either attains its rank or has an $\aleph_\nu$ -node.

(β_2) Every branching table $T = \langle \mathcal{E}, < \rangle$ of rank ω_ν either attains its rank or satisfies the condition: there exists a disjunctive $\aleph_\nu$ -subset $U \subset \mathcal{E}$ not satisfying condition i_ν .

Obviously, $(\alpha_1) \Rightarrow (\alpha_2)$. Theorems proved with the aid of the hypotheses (α_1) and (α_2) will be marked by $(*)$ and $(**)$, respectively.

$(*)$ For every $\aleph_\nu$ -set $U \subset \mathcal{E}$ in the table $T = \langle \mathcal{E}, < \rangle$ of rank $\leq \omega_\nu$, I_ν or II_ν is valid.

2. By the Π^τ -product of discrete spaces $\{X_\alpha : \alpha < \omega_\tau\}$, denoted by us as $\Pi^\tau X_\alpha$ and called a generalized Baire space, we shall mean the space on the abstract product

$$\prod_{\alpha < \omega_\tau} X_\alpha,$$

whose open-closed base B^τ is given by the totality of generalized Baire intervals $\delta_{(i_\beta)_{\beta < \alpha}}$, where

$$\delta_{(i_\beta)_{\beta < \alpha}} = \left\{ (j_\beta)_{\beta < \omega_\tau} \in \prod_{\beta < \omega_\tau} X_\beta : (\forall \beta < \alpha) [j_\beta = i_\beta] \right\}.$$

We shall assume

$$\delta_{(i_\beta)_{\beta < 0}} = \prod_{\alpha < \omega_\tau} X_\alpha.$$

If $(\forall \alpha < \omega_\tau) [X_\alpha = J]$ and $J \neq \aleph_\tau$, then the space $\Pi^\tau X_\alpha$ is denoted by J^{ω_τ} . Put $T^\tau = \langle B^\tau, < \rangle$, where as the relation $<$ the relation $\supset$ of strict inclusion is taken.

$(*)$ In order that a closed set $E \subseteq \Pi^\tau X_\alpha$ not be $\aleph_\nu$ -bicomact, it is necessary and sufficient that it have a disjunctive

an $\aleph_\nu$ -covering $S \subset B^\nu$, satisfying condition Π_ν , i.e., such that the branching table $\langle S(T^\nu), < \rangle$ has an $\aleph_\nu$ -node.

Remark. I. I. Parovichenko ⁽⁴⁾ proved that hypothesis (a_1) is equivalent to the assertion of $\aleph_\nu$ -bicomactness of the T^ν -product of discrete spaces $\{X_\alpha : \alpha < \omega_\nu\}$, where $(\forall \alpha < \omega_\nu) [\overline{X}_\alpha < \aleph_\nu]$. Since the classes of open sets in this space and in the space $\prod^\nu X_\alpha$ coincide, hypothesis (a_1) is equivalent to the assertion of $\aleph_\nu$ -bicomactness of the space $\prod^\nu X_\alpha$, when $(\forall \alpha < \omega_\nu) [\overline{X}_\alpha < \aleph_\nu]$.

3. The set of all tuples of the form $(i_\beta)_{\beta < \alpha}$, corresponding to all points $x \in J^\nu$, will be denoted by W . A set $U \subset W$ will be called a W -base. Unless special qualifications are made, the R - and R^c -operations are considered with the full depth of chains $v = \omega_\nu$. The rigid W -bases of the operations $R_{\mathfrak{M}}$, $R_{\mathfrak{N}}^c$, R_N^α , $R_N^{\alpha c}$, R^α , $R^{\alpha c}$, $0 \leq \alpha < \omega_{\nu+1}$ ^(5, 6) will be denoted respectively by $\theta_{\mathfrak{M}}$, $\theta_{\mathfrak{M}}^c$, θ_N^α , $\theta_N^{\alpha c}$, θ^α , $\theta^{\alpha c}$. By χ_α $[\chi_\alpha^c]$ we denote the rigid base of the projective A_α $[CA_\alpha]$ -operation, $0 \leq \alpha < \omega_{\nu+1}$.

4. We shall say that a rigid base N admits a V -transformation if, for every $J' \subset J$, under the condition $N^{J'} \neq \emptyset$, there exists a set $[J']$ such that, putting for an arbitrary system of sets (E_i) $E_i = E'_i$, if $i \notin [J']$, and $E_i = \emptyset$, if $i \in [J']$, we obtain $\Phi_N^{J'}(E_i) = \Phi_N(E_i)$.

We have proved: if N and N^c are rigid bases of mutually complementary operations, then each of them admits a V -transformation.* Therefore the bases χ_α , χ_α^c , $0 \leq \alpha < \omega_{\nu+1}$, and also the bases θ_N^α , $\theta_N^{\alpha c}$, $0 \leq \alpha < \omega_{\nu+1}$, under the condition that N and N^c are rigid bases, admit a V -transformation.

A rigid base N is called $\aleph_\tau$ -regular ⁽⁸⁾ for a class of sets $\mathcal{K}$ if $(\forall J' \subset J)[\Phi_N^{J'}(\mathcal{K}) \subset \Phi_N(\mathcal{K})]$ and the class of sets $\Phi_N(\mathcal{K})$ is invariant with respect to the operations $\bigcup_{\aleph_\tau}$, $\bigcap_{\aleph_\tau}$. A property H of chains of a rigid base N is called

N -regular ⁽⁸⁾ for a class of sets $\mathcal{K}$ if the operations Φ_{HN} , $\Phi_{(HN)^i}$ are weaker than the operation Φ_N with respect to the class of sets $\mathcal{K}$.

Each of the bases χ_α , χ_α^c , $0 \leq \alpha < \omega_{\nu+1}$, and also each of the bases θ_N^α , $\theta_N^{\alpha c}$, $1 \leq \alpha < \omega_{\nu+1}$, when the conditions are fulfilled: 1°. N and N^c are rigid bases; 2°. The operation Φ_N is stronger than the operation $\bigcup_{\aleph_\nu}$, is $\aleph_\nu$ -regular for the class of sets $\mathcal{K} \ni \emptyset, \Xi$, where Ξ is the basic space.

Hence, and by virtue of Theorem 3 of I. Kozlova ⁽⁸⁾, Theorem 1), for the class of sets $\mathcal{K} \ni \emptyset, \Xi$, for every $p \leq \aleph_0$ the property H_x is M -regular, if M is a base χ_α , χ_α^c , $0 \leq \alpha < \omega_{\nu+1}$, and also a base θ_N^α , $\theta_N^{\alpha c}$, $1 \leq \alpha < \omega_{\nu+1}$ in the case when conditions 1° and 2° are fulfilled, and, consequently, when M is a base θ^α , $\theta^{\alpha c}$, $1 \leq \alpha < \omega_{\nu+1}$.

5. For a base N define the $\Delta\Sigma$ -operation Q_N over two systems of sets (E_u) , (e_v) , by putting

$$Q_N(E_u, e_v) = \bigcup_{u,v} \bigcup_{\xi \in N, \xi' \subset \xi, \bar{\xi}' = \aleph_\nu} \left(\bigcap_{u \in \xi} E_u \cap \bigcap_{v \in \xi'} e_v \right).$$

(*) For any $0 \leq \alpha < \omega_{\nu+1}$ the operation Q_{θ^α} [$Q_{\theta^{\alpha c}}$] is weaker than the operation Φ_{θ^α} [$\Phi_{\theta^{\alpha c}}$] with respect to the class of sets $\mathcal{K} \ni \emptyset, \Xi$.

6. Let $\mathfrak{M} = (N_a)_{a \in W}$, $\mathfrak{M}^c = (N_a^c)_{a \in W}$ be tables of rigid bases. For the operations $\Phi_{\theta\mathfrak{M}}$, $\Phi_{\theta\mathfrak{M}^c}$, we introduce thinning conditions. We shall say that a collection of tuples η_x satisfies condition a_1^ν [a_3^ν] if in the branching table $\langle \eta_x(T_W^\nu), < \rangle$ there is a node $((a_i))_{i \in J'}$ from which one can form a $> \aleph_\nu$ [$\geq \aleph_\nu$] R -(R^c -) covering of the tuple a ; a_2^ν , if in the collection $\eta_x(T_W^\nu)$ there is a tuple and such an R -(R^c -) covering of it in which $\aleph_\nu$ tuples have the property of bisection; a_4^ν , if the table $\langle \mu_x, < \rangle$ attains its rank ω_ν ; a_7^ν , if the collection μ_x includes a disjunctive $\aleph_\nu$ -subset of tuples.

* A. D. Taimanov ⁽⁷⁾ gave an example of a δS -operation with a rigid base for which the complementary δS -operation has no rigid base.

Then for the operations $\Phi_{\theta_{\mathfrak{M}}}, \Phi_{\theta_{\mathfrak{M}}^c}$ the following holds:

1) for $\nu < \omega_\nu$:

$$\overline{M}_x > \mathfrak{N}_\nu \iff a_1^\nu \vee a_2^\nu, \quad \overline{M}_x \geq \mathfrak{N}_\nu \iff a_2^\nu \vee a_3^\nu, \quad \overline{M}_x = \mathfrak{N}_\nu \iff \neg(a_1^\nu \vee a_2^\nu) \& a_3^\nu;$$

2) (**)

$$\overline{M}_x \geq \mathfrak{N}_\nu \& \neg a_3^\nu \& \neg a_7^\nu \Rightarrow a_4^\nu;$$

3) (**).

$$\overline{\overline{M}}_x \geq \mathfrak{N}_\nu \iff a_3^\nu \vee a_4^\nu \vee a_7^\nu.$$

For the operation $\Phi_{\theta_{\mathfrak{M}}^c}$ the following holds:

4) (*)

$$\overline{\overline{M}}_x > \mathfrak{N}_\nu \iff a_1^\nu \vee a_2^\nu;$$

5) (*)

$$\overline{\overline{M}}_x = \mathfrak{N}_\nu \iff \neg(a_1^\nu \vee a_2^\nu) \& a_3^\nu.$$

7. Let the W -base U coincide with the base $\theta_{\mathfrak{M}}$ or $\theta_{\mathfrak{M}}^c$. For $k = 2, 4, 7$ put

$$\Phi_{H_{a_k} U}(E_a) = \{x \in \Phi_U(E_a) : \text{the set } \eta_x \text{ satisfies the condition } a_k^\nu\}.$$

(*) For the class of sets $\mathcal{K} \ni \emptyset, \Xi$, the operations

$$\Phi_{H_{a_2} \theta^\alpha}, \quad \Phi_{H_{a_2} \theta^{\alpha c}}, \quad 0 \leq \alpha < \omega_{\nu+1},$$

are weaker than the operations $\Phi_{\theta^\alpha}, \Phi_{\theta^{\alpha c}}$, respectively.

(**) If conditions 1⁰ and 2⁰ are fulfilled, the operations

$$\Phi_{H_{a_4} \theta_N^\alpha}, \quad \Phi_{H_{a_7} \theta_N^\alpha}, \quad 2 \leq \alpha < \omega_{\nu+1},$$

are weaker than the operation $\Phi_{\theta_N^\alpha}$, and the operations

$$\Phi_{H_{a_4} \theta_N^{\alpha c}}, \quad \Phi_{H_{a_7} \theta_N^{\alpha c}}, \quad 1 \leq \alpha < \omega_{\nu+1},$$

are weaker than the operation $\Phi_{\theta_N^{\alpha c}}$.

(**) If conditions 1⁰ and 2⁰ are fulfilled, the property $H_{\mathfrak{N}_\nu}$ is M -regular if the base M coincides with the base θ_N^α , $2 \leq \alpha < \omega_{\nu+1}$, $\theta_N^{\alpha c}$, $1 \leq \alpha < \omega_{\nu+1}$, in particular, with the base θ^α for $2 \leq \alpha < \omega_{\nu+1}$, and $\theta^{\alpha c}$ for $1 \leq \alpha < \omega_{\nu+1}$.

(*) The property $H_{\mathfrak{Z}_\nu}$ is $\theta^{\alpha c}$ -regular for the class of sets $\mathcal{K} \ni \emptyset, \Xi$.

8. Let $J = \{\alpha : \alpha < \omega_\nu\}$. If each chain ξ of the rigid base M is ordered in the order of increase of its elements, then we obtain a rigid reduced base $\check{M}$. Let $\check{M} \subset J^{\omega_\nu}$. Put

$$\Phi_{H_c \check{M}}(E_i) = \{x \in \Phi_M(E_i) : \text{the closure of the set } \check{M}_x \text{ is not } \mathfrak{N}_\nu\text{-bicomact}\},$$

where $\check{M}_x$ is the set of all chains of the base $\check{M}$ whose kernels contain the point x . Let

$$q = (i_\beta)_{\beta < \alpha} \in W \quad \text{and} \quad (\beta' < \beta'') \Rightarrow (i_{\beta'} < i_{\beta''}).$$

For a system of sets (E_i) put $E_i^q = \emptyset$, if

$$(\forall \beta < \alpha)(i \neq i_\beta) \ \& \ (\exists \beta_0)[i < i_{\beta_0}],$$

and $E_i^q = E_i$ in all remaining cases. Let

$$\mathcal{E}_q = \Phi_M(E_i^q), \quad S'_\alpha = \bigcup_{q \in W} \mathcal{E}_{qp}.$$

Then

$$(*) \quad \Phi_{H_c \check{M}}(E_i) = \bigcup_{q \in W} \overline{\lim}_i^{\mathfrak{N}_\nu} S_{qi}.$$

(*) If the rigid base M is $\mathfrak{N}_\nu$ -proper for the class of sets $\mathcal{K} \ni \emptyset$, then the property H_c is M -regular.

(*) The property H_c is M -regular for the class of sets $\mathcal{K} \ni \emptyset, \Xi$, if the base M coincides with the base $\chi_\alpha, \chi_\alpha^c$, $0 \leq \alpha < \omega_{\nu+1}$, and also, when conditions 1^0 and 2^0 are fulfilled, with the base $\theta_N^\alpha, \theta_N^{\alpha c}$, $1 \leq \alpha < \omega_{\nu+1}$, in particular, with the base $\theta^\alpha, \theta^{\alpha c}$, $0 \leq \alpha < \omega_{\nu+1}$.

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- ¹ I. D. Stupina, DAN, **188**, No. 5, 1010 (1969).
- ² I. D. Stupina, DAN, **117**, No. 2, 188 (1957).
- ³ G. Kurera, Publ. math. de l' Univ. de Belgrade, **4**, 1 (1935).
- ⁴ I. I. Parovichenko, DAN, **174**, No. 1, 30 (1967).
- ⁵ A. A. Lyapunov, Algebra and Logic, Seminar, **2**, No. 2, 47 (1963).
- ⁶ Z. I. Kozlova, Scientific Notes, Volgograd Ped. Inst., Mathematics, issue 23 (1969).

⁷ A. D. Taimanov, Izv. AN SSSR, ser. matem., **14**, 443 (1950).

⁸ Z. I. Kozlova, DAN, **188**, No. 5, 1001 (1969).

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