

ON MAPPINGS IN HILBERT SPACE

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Abstract

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MATHEMATICS

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ON MAPPINGS IN HILBERT SPACE

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The principal aim of the present work is to extend the Leray-Schauder degree theory to a class of mappings M in Hilbert space. The definition of the class M is given in § 2. Mappings of this class are naturally called almost-monotone mappings. In particular, the class M includes mappings of the form $\varphi + \Phi$, where φ is a monotone mapping and Φ is a completely continuous mapping. A number of fixed-point theorems and a theorem on invariance of domain are presented.

1. Below, H always denotes a real Hilbert space. All mappings under consideration are assumed to be continuous. By I we denote the identity mapping, and by $\Phi(\Phi_i)$ completely continuous mappings.

Since one has to consider mappings of the class M that depend continuously on a parameter, it is convenient to give the definition of the class M when the domain is a closed set in the space $E = H \times R$, where R is a compact metric space. We shall usually put $R = [0, 1]$. The symbol E will always denote a space of the indicated form. Points of E will be denoted by the letter z , and points of H by the letter x . Replacing, in those definitions and propositions in which E occurs, the letter z by x and E by H , we obtain the corresponding definition or proposition for H . This makes it possible to shorten the corresponding formulations.

2. **Definition 1.** Let A be a bounded closed set in E , and let f be a mapping of A into H . We shall say that $f \in M(A)$ if the following conditions are satisfied: 1) $f(A)$ is a bounded set; 2) if A is a noncompact set, then for every noncompact set $B \subset A$

$$\limsup_{z, z' \in B} (f(z) - f(z'), x - x') > 0,$$

where $z = (x, t)$, $z' = (x', t')$, $x, x' \in H$, $t, t' \in R$.

Thus, if A is a bounded noncompact closed set in the Hilbert space H , then condition 2) means that for every noncompact set $B \subset A$

$$\limsup_{x,y \in B} (f(x) - f(y), x - y) > 0.$$

In Hilbert space, the class $M(A)$ includes mappings of the form $\varphi + \Phi$, where φ is a strictly monotone mapping, i.e. a mapping satisfying the condition

$$(\varphi(x) - \varphi(y), x - y) \geq \alpha(r) > 0, \quad r = \|x - y\|.$$

A strictly monotone mapping is a mapping of the form $I - K$, where K satisfies the condition

$$(K(x) - K(y), x - y) \leq q(r)\|x - y\|^2, \quad r = \|x - y\|, \quad q(r) < 1.$$

Definition 2. Let A be a bounded closed set in E , and let $f_1, f_2 \in M(A)$. We shall say that f_1 is homotopic to f_2 in the class M , if

there exists a homotopy $g(z, t)$, $z \in A$, $0 \leq t \leq 1$, joining f_1 and f_2 , such that $g(z, t)$ is a mapping of class $M(C)$, $C = A \times [0, 1]$. The corresponding homotopy is called a homotopy in the class M .

Below, in Proposition 1, $L(\lambda)$ denotes the class of mappings $E \rightarrow H$ representable in the form $\lambda(z)x + \Phi(z)$, $z = (x, y)$, where $\lambda(z)$ is a positive function satisfying the condition $0 < c_1 < \lambda(z) < c_2 < \infty$.

Proposition 1. Let A be a bounded closed set in E . Then: 1) $M(A)$ is a convex set. 2) If $f \in M(A)$ and g_λ, g_μ are mappings of class $L(\lambda)$, then the mapping $g_\lambda f g_\mu + \Phi$ is also a mapping of class $M(A)$. 3) If $f \in M(A)$, then $f(A)$ is a closed set and, for any compact set F , the set $f^{-1}(F) \cap A$ is compact.

3. By $T(H)$ we denote the set of all finite-dimensional subspaces of H . $T(H)$ is ordered by inclusion. If $T \in T(H)$, then by DT we denote the orthogonal complement of T , and by p_T (p_{DT}) the orthogonal projection onto T (respectively onto DT).

Let G be a bounded open set in H , $\bar{G}$ its closure, Γ the boundary of G , and f a continuous mapping $\bar{G} \rightarrow H$. For any finite-dimensional subspace T , by $G(T)$ we denote $G \cap T$, and by $\Gamma(T)$ the boundary in T of the open set $G(T)$. If $G(T)$ is nonempty, then by f_T we denote the mapping $\bar{G(T)} \rightarrow T$, defined as follows: $f_T(x) = p_T f(x)$, $x \in T \cap \bar{G}$ (p_T is the orthogonal projection onto T). If $a \in T \setminus f_T(\Gamma \cap T)$, then the degree of the mapping f_T , considered on the set $G \cap T$, at the point a is defined; we denote it by $c(f_T, a, G \cap T)$.

Let a be a point in H such that a lies outside $f(\Gamma)$. We shall say that f (considered on G) has mapping degree $c(f, a, G)$ at the point a over Z , if the following conditions are satisfied:

- a) There exists a $T_0 \in T(H)$, containing a , such that for any $T \supseteq T_0$ the set $f_T(\Gamma \cap T)$ does not contain a , i.e. for any $T \supseteq T_0$ the value $c(f_T, a, G \cap T)$ is defined.
- b) The values $c(f_T, a, G \cap T)$ stabilize over Z , i.e. there exists a $T_1 \in T(H)$ such that $c(f_T, a, G \cap T) = c(f, a, G)$, if $T \supseteq T_1$.

We note that if condition 1) is fulfilled, then stabilization of $c(f_T, a, G \cap T)$ takes place modulo 2, but, generally speaking, there may be no stabilization over Z ; see ⁽²⁾.

Below, to the end of the note, by mapping degree we mean mapping degree over Z .

- 4. For mappings of class M , the mapping degree is defined at all points lying outside the image of the boundary, and computation of the mapping degree reduces to computation of the mapping degree of a mapping of the form $I - \Phi$. More precisely, the following holds.

Proposition 2. *Let G be a bounded open set in H , Γ the boundary of G , $f: \bar{G} \rightarrow H$, and $f \in M(\bar{G})$. Then $f(\Gamma)$ is a closed set, and for any point a lying outside $f(\Gamma)$, there exists a neighborhood U in H and a $T_0 \in T(H)$ such that: 1) for any point $b \in U$ the mapping degree $c(f, b, G)$ is defined, and this value is constant on U ; 2) for any $b \in U$ and any $T \supseteq T_0$, $c(f, b, G)$ coincides with the degree of the mapping $I + p_T(f - I)$ at the point b .*

In the class M , the mapping degree has all the basic properties of the mapping degree (see, for example, ⁽¹⁾, § 1): 1) additivity; 2) $a \in f(G)$, if $c(f, a, G) \neq 0$; 3) the degree $c(f, a, G)$ remains constant under a continuous deformation of the domain G and of the mapping f (in the class M), if a lies outside the image of the boundary under this deformation.

For mappings of class M the following classification theorem holds.

Proposition 3. *Let G be a bounded open set in H ; Γ the boundary of G ; f_1, f_2 mappings of class $M(\bar{G})$. Suppose $f_2(\Gamma) \cup f_1(\Gamma)$ does not contain the point a , $c_1 = c(f_1, a, G)$, $c_2 = c(f_2, a, G)$. The equality $c_1 = c_2$ has*

takes place if and only if there exists a homotopy $g(x, t)$ in the class M , connecting f_1 and f_2 , for which $\inf \|g(x, t) - a\| > 0$, $x \in \Gamma$, $0 \leq t \leq 1$.

Let A be a bounded closed set in E , f a mapping of A into H of the class $M(A)$, $T \in T(H)$. Define the mapping $\psi[f, T]$ as follows:

$$\psi[f, T](x, t) = f(x) + tp_{DT}(x - f(x)), \quad x \in A, \quad 0 \leq t \leq 1.$$

The mapping $\psi[f, T]$ is defined on $A \times [0, 1]$ and is a mapping of the class M . We have $\psi[f, T](x, 0) = f(x)$ and $\psi[f, T](x, 1) = x + p_T[f(x) - x]$, i.e., if t is regarded as a deformation parameter, then $\psi[f, T]$ is a deformation in the class M taking f into the mapping $I + p_T(f - I)$.

The proof of the propositions stated above is based on the following proposition:

Proposition 4. Let A be a bounded closed set in E , f a mapping of A into H of the class $M(A)$. For every point a lying outside $f(A)$, there exists a $T_0 \in T(H)$, containing the point a , and an $\varepsilon > 0$ such that, for every $T \supset T_0$, $\|p_T \psi(f, T_0)(x, t) - a\| > \varepsilon$, if $x \in T \cap A$, $0 \leq t \leq 1$.

5. Let A be a closed bounded set in E . By $M_0(A)$ we denote the set of continuous mappings $A \rightarrow H$ which is the closure, in the uniform metric, of the set $M(A)$. If A lies in H , then this is equivalent to saying that $f(A)$ is a bounded set and, for every noncompact set $B \subset A$,

$$\limsup_{x, y \in B} (f(x) - f(y), x - y) \geq 0.$$

The class M_0 includes, for example, mappings of the form $I - K - \Phi$, where $\|K(x) - K(y)\| \leq \|x - y\|$. Let G be a bounded open set in H ; Γ the boundary of G ; $f : \overline{G} \rightarrow H$. If $f \in M_0(\overline{G})$, then the set $f(\Gamma)$, generally speaking, may be nonclosed, and the degree of the mapping f is defined at all points lying outside $\overline{f(\Gamma)}$ in the following way.

For every point a lying outside $\overline{f(\Gamma)}$, there exist $\lambda(a) > 0$ and a neighborhood $U(a)$ such that, for every λ satisfying $0 < \lambda < \lambda(a)$, the mapping $f_\lambda = f + \lambda I$ has a degree of mapping at all points of $U(a)$, and the degree of this mapping does not depend on λ . This value is taken to be $c(f, a, G)$.

Replacing M by M_0 in Definition 2, we obtain the definition of homotopy in the class M_0 . In the class M_0 the classification theorem is also valid (Proposition 3); it is only necessary, instead of $f_i(\Gamma)$, to write everywhere $\overline{f_i(\Gamma)}$.

6. For mappings of the class M the theorem on invariance of domain is valid:

Proposition 5. Let G be an open bounded set in H , f a homeomorphic mapping of G into H . If, for every closed set $A \subset G$, $f \in M(A)$, then $f(G)$ is an open set.

Below we give a number of propositions on fixed points.

Proposition 6. Let Q be an open bounded convex set in H ; Γ the boundary of Q ; $f : \overline{Q} \rightarrow H$. Let $f \in M(\overline{Q})$, and let a be a point of Q such that the equation $f(x) = \lambda(x - a)$ has no solutions for $x \in \Gamma$ and any $\lambda \geq 0$ (or for any $\lambda \leq 0$). Then $f(Q)$ contains the point a .

From Proposition 6 it follows immediately:

Proposition 7. Let Q be an open bounded convex set in H ; Γ the boundary of Q ; f such a mapping $\overline{Q} \rightarrow H$ that $I - f$ is a mapping of the class $M(\overline{Q})$. Then f has a fixed point in $\overline{Q}$, if one of the two conditions is satisfied: a) $f(\Gamma) \subseteq \overline{Q}$; b) $\overline{Q}$ is a ball and $(f(x), x) \neq (x, x)$ for $x \in \Gamma$.

The following proposition generalizes the Borsuk-Ljusternik-Schnirelman theorem:

Proposition 8. Let S be a sphere in H with center at the point O ; let f be a mapping of class $M(S)$ such that the equation $f(x) = \lambda f(-x)$ has no solutions on S for any $\lambda \geq 0$. Then the degree of the mapping f with respect to the point O is odd.

Proposition 9 is a generalization of Brouwer's theorem ⁽³⁾.

Proposition 9. Let f be a mapping of H into H such that the mapping $I - f$ is a mapping of class $M(A)$ for any closed set A . If, for some positive integer m , the iteration $f^m(H)$ is bounded, then f has a fixed point.

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