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Abstract

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MATHEMATICS

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ON THE OSCILLATION OF THE SEQUENCE OF FEJÉR SUMS

(Presented by Academician S. L. Sobolev on 16 IX 1969)

Let $\sigma_n = \sigma_n(f; x)$ and $\sigma_m = \sigma_m(f; x)$ be the Fejér sums of order m and n ($m, n = 1, 2, \dots$) of a 2π -periodic function $f(x)$. In this paper we consider the problem of finding the least upper bound of the modulus of the difference $\sigma_m - \sigma_n$ on the classes $W^{(p)}$ ($p = 1, 2, 3, \dots$) of 2π -periodic functions that have an absolutely continuous derivative of order $(p-1)$ and a derivative of order p , bounded in modulus by one everywhere where this derivative exists, and also on the corresponding classes of conjugate functions. A similar problem, called the problem of oscillation, for the Abel-Poisson summation method was studied earlier in papers ^(5,6). Let

$$C_p = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(p+1)}}{(2\nu+1)^p}, \quad \tilde{C}_p = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu p}}{(2\nu+1)^p}.$$

Theorem 1. For any two values m, n and $p \geq 2$, the equality

$$\sup_{f \in W^{(p)}} |\sigma_m - \sigma_n| = |\sigma_m(\varphi_p; x_p) - \sigma_n(\varphi_p; x_p)| \quad (1)$$

holds, where $x_p = [1 + (-1)^{p+1}]\pi/4$, and $\varphi_p(x)$ is a function from the class $W^{(p)}$ for which $\varphi_p^{(p)}(x) = \text{sign } \cos x$.

Theorem 2. Uniformly with respect to m, n ($m \geq n$), as $m, n \rightarrow \infty$, the following asymptotic equality holds:

$$\sup_{f \in W^{(p)}} |\sigma_m - \sigma_n| = \begin{cases} \frac{2(m-n)}{\pi mn} [\ln n + O(1)], & p = 1, \\ C_p \frac{m-n}{(m+1)(n+1)} [1 + O(n^{-p+1})], & p > 1. \end{cases} \quad (2)$$

Theorem 3. For any values $p \geq 2$, the equality

$$\sup_{f \in W^{(p)}} |\tilde{\sigma}_m(f; x) - \tilde{\sigma}_n(f; x)| = |\tilde{\sigma}_m(\varphi_p; x_{p+1}) - \tilde{\sigma}_n(\varphi_p; x_{p+1})| \quad (3)$$

holds.

Theorem 4. For any values $p \geq 2$, uniformly with respect to m and n ($m \geq n$), as $m, n \rightarrow \infty$, the following asymptotic equality holds:

$$\sup_{f \in W^{(p)}} |\tilde{\sigma}_m(f; x) - \tilde{\sigma}_n(f; x)| = \frac{\tilde{C}_p(m-n)}{(m+1)(n+1)} [1 + O(n^{-p+1})]. \quad (4)$$

The proof of equality (1) is based on the fact that the difference $\sigma_m - \sigma_n$ admits a representation of the form

$$\frac{1}{\pi} \int_0^{2\pi} f^{(p)}(x+t) K(t) dt,$$

where

$$K(t) = \sum_{k=1}^n \frac{m-n}{(m+1)(n+1)k} \frac{\sin kt}{k} + \sum_{k=n+1}^m \left(\frac{1}{k^2} - \frac{1}{k(m+1)} \right) \frac{\sin kt}{k}. \quad (5)$$

Abel's transformation makes it possible to show that expression (5) is a linear combination (with positive coefficients) of the functions

$$S_k(t) = \sum_{\nu=1}^k \frac{\sin \nu t}{\nu},$$

which are positive on $(0, \pi)$ (see (2), p. 106). Therefore the odd function $K(t)$ preserves its sign on $(0, \pi)$.

Equality (3) in the case $p \geq 3$ follows from (1), since it can be shown that the right-hand side of (1) is equal to

$$-\frac{4}{\pi(m+1)} \left| \frac{m-n}{n+1} \sum_{\nu=0}^{[(n-1)/2]} \frac{(-1)^{\nu(p+1)}}{(2\nu+1)^p} + \sum_{\nu=[(n-1)/2]+1}^{[(m-1)/2]} \frac{(-1)^{\nu(p+1)}(m-2\nu)}{(2\nu+1)^{p+1}} \right|.$$

In the case $p = 1, 2$, taking into account the equalities (see (2), p. 86)

$$\sum_{\nu=1}^k \sin \nu t = \frac{1 - \cos kt}{2 \operatorname{tg} t/2} + \frac{1}{2} \sin kt,$$

we obtain

$$\sigma_n - \sigma_m = \frac{1}{\pi} \int_0^{2\pi} f'(x+t)M(t) dt,$$

where

$$M(t) = \sum_{k=n+1}^m \frac{1 - \cos kt}{2k(k+1) \operatorname{tg} t/2} + \frac{1}{2} \sum_{k=n+1}^m \frac{\sin kt}{k(k+1)}.$$

The first sum in the preceding formula is an odd function, positive on $(0, \pi)$. Using this fact, it is not difficult to conclude that in the case $p = 1, 2$ the left-hand side of (1), up to the remainder term in equality (2), is equal to the right-hand side of (1). Thus, in this case as well, the proof of equality (2) is reduced to the computation of the right-hand side of (1).

The proof of Theorems 3 and 4 is carried out according to the same scheme. Estimates (2) and (4) are a generalization of the known results of S. M. Nikol'skii (4), which are obtained from these estimates by passing to the limit as $m \rightarrow \infty$.

Relations analogous to (1)–(4) hold also in the case when one considers the oscillations of the sequence of Fejér sums in the L -metric on classes with p -th derivative bounded in the L -metric.

Remark 1. Equality (1) for $p = 1, 2$ does not hold in general. However, for $p = 2$ it is valid for every n and all values of m , beginning with some value $m_0(n)$ depending on n . In the case $p = 1$, such an assertion is valid only for even values of n .

Remark 2. With the aid of the methods used to obtain the preceding results, one can show that on the functions $\varphi_p(x)$ the upper bound is also attained in problems on the oscillation of the sequence of de la Vallée-Poussin integrals

$$V_n = \frac{(2n)!!}{2\pi(2n-1)!!} \int_0^{2\pi} f(x+t) \cos^{2n}(t/2) dt$$

(on the classes $W^{(p)}$, $p \geq 1$); Jackson integrals

$$I_n = \frac{3/2(n+1)^{-1}}{\pi[2(n+1)^2+1]} \int_0^{2\pi} f(x+t) \frac{\sin^4(n+1)t/2}{\sin^4 t/2} dt,$$

Cesàro means $\sigma_n^{(\alpha)}(f; x)$ ($\alpha > 1$), and Bernstein-Rogozinskii sums

$$B_n = \frac{1}{2} [S_n(f; x + \pi/2n) + S_n(f; x - \pi/2n)]$$

on the classes $W^{(p)}, p \geq 3$.

We shall give some asymptotic estimates related to this. Uniformly with respect to m and n ($m \geq n \geq 1$), the following relations hold:

$$\sup_{f \in W^{(1)}} |V_m - V_n| = \frac{4}{\pi} \left\{ \frac{(2n)!!}{(2n+1)!!} - \frac{(2m)!!}{(2m+1)!!} \right\} + O\left(\frac{m-n}{mn}\right), \quad (6)$$

$$\sup_{f \in W^{(p)} (p \geq 3)} |I_m - I_n| = C(m, n, p) [1 + O(n^{-1} + n^{-p+2} \ln n)], \quad (7)$$

$$\sup_{f \in W^{(p)} (p \geq 3)} |B_m - B_n| = \frac{1}{12} \pi^2 C(m, n, p) [1 + O(n^{-2} + n^{-p+2})], \quad (8)$$

where $C(m, n, p) = \frac{3}{2}(m^2 - n^2)m^{-2}n^{-2}C_{p-1}$.

In the asymptotic estimate for the oscillations of the sequence of Cesàro means $\sigma_n^{(\alpha)}(f; x)$ ($\alpha > 2$) on the classes $W^{(p)}$ ($p \geq 1$), the leading term differs by the factor α from the leading term in estimate (2).

Equalities (6) and (8), with a somewhat different form of the remainder term in the case $m = \infty$, were obtained earlier in the works, respectively, ⁽³⁾ and ⁽¹⁾.

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