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Abstract

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MATHEMATICS

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ON SOME PROPERTIES OF SPACES WITH BICOMPACT REMAINDERS OF FINITE ORDERS

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1. In this paper we study some properties of spaces of Iséki orders n , i.e. of the classes K_n ($n \geq -1$), and of their mappings. The spaces are assumed to be completely regular and Hausdorff, and the mappings continuous.

As is known ⁽¹⁾, a space X belongs to the class K_n ($n \geq 0$) in the sense of Iséki if and only if $R^{n/2}(X)$ is bicomact when n is even, and $R^{(n-1)/2}(X)$ is locally bicomact when n is odd. $R^i(X)$ is defined as follows: $R^0(X) = X$, $R^1(X)$ is the set of all points of non-local bicomactness of X , and $R^i(X) = R^1[R^{i-1}(X)]$ for $i > 1$.

In ⁽¹⁾ an "external" characteristic of the spaces of the classes K_n is also given: a space X belongs to the class K_n if and only if the remainder of order n ⁽¹⁾ is bicomact. It is clear that K_0 coincides with the class of bicomacts, and K_1 with the class of locally bicomact spaces. We shall further assume that the class K_{-1} consists of empty spaces.

In what follows another characteristic of the classes K_n will be needed.

Theorem 1. *A space X belongs to the class K_n ($n \geq 1$) if and only if $R^1(X)$ belongs to the class K_{n-2} .*

It is shown in ⁽¹⁾ that a closed subset of a space from the class K_n also belongs to the class K_n . For open subsets the following holds.

Theorem 2. *Let X be a space of the class K_n , and let $G \subseteq X$ be an open subset of it. Then G belongs to the class K_{n+1} if n is even, and to the class K_n if n is odd.*

It is known that local bicomactness is preserved under open mappings. For spaces from the classes K_n the following holds.

Theorem 3. *The open image of a space of the class K_n belongs to the same class K_n .*

In ⁽¹⁾ the question of the topological product of spaces from the classes K_n was considered in the special case when one of the factors is either bicomact or locally bicomact. In the general case the following holds.

Theorem 4. *Let X belong to the class K_m , and Y to the class K_n ; then $X \times Y$ belongs to the class K_{m+n} if either m or n is even, and to the class K_{m+n-1} if m and n are odd.*

The next theorem gives an internal characteristic of a space with a bicomact remainder of finite order, i.e. of some class K_n , and is an answer to a question posed by G. S. Chogoshvili.

Theorem 5. *A space X has a bicomact remainder of finite order if and only if it can be decomposed into the sum of a finite number of pairwise disjoint subsets, each of which is locally bicomact in the induced topology.*

For closed mappings of spaces from the classes the following theorems hold:

Theorem 6. *Let X be a hereditarily weakly paracompact space and let $f : X \rightarrow Y$ be a closed mapping. Then, if X belongs*

class K_n , then Y belongs to the class K_{2n} for even n and to the class K_{2n+1} for odd n .

Theorem 7. *Let X be a hereditarily weakly paracompact space and $f : X \rightarrow Y$ a closed mapping such that $R(f)$ (see (1)) is bicomact. Then, if X belongs to the class K_n , then Y belongs to the class K_n for even n and to the class K_{n+1} for odd n .*

The following theorem is a strengthening of a theorem of Lelek from ⁽²⁾.

Theorem 8. *Let X be a finally compact space of countable type and let $f : X \rightarrow Y$ be a mapping such that $f^{-1}(y)$ belongs to K_n for every $y \in Y$. Then*

$$\dim X \leq \dim f(X) + \max\{\dim f, \text{def } X\}.$$

Definition 1. A space X is called **n -peripheral** if in this space there exists a base of open sets whose boundaries belong to the class K_n .

The following theorem may be regarded as a generalization of the Freudenthal-Morita theorem ⁽³⁾, obtained for $n = -1$.

Theorem 9. *Let X be a space having a base of open sets $U = \{U_\alpha\}_{\alpha \in M}$ with the following properties: 1) the boundary $\text{Fr } U_\alpha$ belongs to the class K_n for every $\alpha \in M$; 2) if $U_\alpha \in U$, then $X \setminus [U_\alpha] \in U$; 3) for every closed $A \subseteq U_\alpha \in U$ there exists $U_\beta \in U$ such that $A \subseteq [U_\beta] \subseteq U_\alpha$; 4) if $U_\alpha \in U$ and $U_\beta \in U$, then $U_\alpha \cup U_\beta \in U$. Then there exists a bicomact extension bX with an $(n - 1)$ -peripheral remainder $bX \setminus X$.*

Definition 2. A base $U = \{U_\alpha\}_{\alpha \in M}$ of open sets of X is called **regular** if the following conditions are satisfied: 1) if $U_\alpha \in U$ and $U_\beta \in U$, then $U_\alpha \cup U_\beta \in U$; 2) if $U_\alpha \in U$, then $X \setminus [U_\alpha] \in U$; 3) if $U_\alpha \in U$ and $U_\beta \in U$, then $U_\alpha \setminus [U_\beta] \in U$.

If the Freudenthal-Morita theorem gives conditions under which there exists an extension with a zero-dimensional first remainder, then the following theorems give conditions for the existence of extensions with zero-dimensional remainders of the 2nd and 3rd orders. Zero-dimensionality is understood in the sense of ind.

Theorem 10. Let X be a paracompact 1-peripheral space. Then there exists an extension of order 2 with a zero-dimensional remainder.

Theorem 11. Let X be a paracompact space of countable type and let $U = \{U_\alpha\}_{\alpha \in M}$ be the collection of all open subsets of X whose boundaries are bicompatly situated in the sense of Chogoshvili ⁽⁴⁾ and belong to the class K_2 . If U is a regular base, then there exists an extension of order 3 with a zero-dimensional remainder.

The following theorem gives a condition under which spaces of the classes K_n are complete in the sense of Čech.

Theorem 12. Let X be a completely normal space of class K_n which is completely normally embedded ⁽⁵⁾ in some bicompat extension. Then X is complete in the sense of Čech.

Corollary. A metrizable space of class K_n is an absolute G_δ .

Remark 1. For arbitrary completely normal spaces Theorem 12 is not true; there even exists a countable completely normal space of class K_2 which is not complete in the sense of Čech. However, the following holds.

Theorem 13. A countable space of class K_n is complete in the sense of Čech if and only if it is metrizable.

There exists a space of class K_n which is not a k -space. We give one sufficient condition under which a space of class K_n is a k -space.

Theorem 14. Let X be a space of class K_n such that for every bicompatum Φ from $R^1(X)$ there exists a bicompatum $F \supseteq \Phi$ of countable character in X . Then X is a k -space.

2. In the survey article ⁽⁶⁾ A. V. Arhangel'skii posed the following problems:

Problem 1. Let X be a finally compact space and $f : X \rightarrow Y$ its closed mapping. Is the set

$$\Phi = \{y : Y \text{ } f^{-1}(y) \text{ is not bicompat}\}$$

always at most countable?

Problem 2. Is every strict p -space perfectly mapped onto some space with a refining sequence of covers?

Below negative solutions of both of the indicated problems are given. Namely, in 2.1 an example will be constructed of a finally compact space of class K_2 in the sense of Isasardze, and of such a closed mapping $f : X \rightarrow Y$ of it that the set Φ is

open, everywhere dense in Y , and has the cardinality of the continuum; in 2.2 an example will be constructed of a locally bicomact, weakly paracompact and σ -paracompact, nonnormal strict p -space which cannot be perfectly mapped onto any space with a refining sequence of covers. In constructing these examples one construction from ⁽⁷⁾ is used, applied there for another purpose.

2.1. Let $OXYZ$ be some rectangular coordinate system in R^3 , and let K^3 be the unit cube:

$$K^3 = E\{(x, y, z) \in R^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}.$$

Consider the opposite faces α and β of the cube K^3 :

$$\alpha = E\{(x, y, z) \in K^3 : 0 \leq x \leq 1, y = 0, 0 \leq z \leq 1\}$$

and

$$\beta = E\{(x, y, z) \in K^3 : 0 \leq x \leq 1, y = 1, 0 \leq z \leq 1\}.$$

Let

$$\beta^* = E\{(x, y, z) \in \beta, z \text{ rational}\}.$$

We introduce a topology on the set $X = \alpha \cup \beta^*$ in the following way: if a point $M \in \beta^*$, then it is isolated in X ; if, however, a point $M \in \alpha$, then its neighborhood is

$$OM = O'M \cup \{P(O'M) \setminus A\},$$

where $O'M$ is an open rectangular neighborhood of the point M in the usual topology of the face α ; $P(O'M)$ is the projection onto β^* ; and A is some countable subset of β^* . It can be shown that X is regular, finally compact, and belongs to the class K_2 . Let

$$\Phi_1 = E\{(x, y, z) \in \alpha : 0 \leq x \leq 1, y = 0, z = 0\}$$

and

$$\Phi_2 = E\{(x, y, z) \in \beta^* : 0 \leq x \leq 1, y = 1, z = 0\}.$$

The set $Y = \Phi_1 \cup \Phi_2 \subseteq X$ is closed in X . Therefore $Y = \Phi_1 \cup \Phi_2$, in the induced topology, also has the listed properties of X . The mapping $f : X \rightarrow Y$ is defined as follows:

$$f(x, y, z) = (x, 1, 0), \quad \text{if } M(x, y, z) \in \beta^*,$$

and

$$f(x, y, z) = (x, 0, 0), \quad \text{if } M(x, y, z) \in \alpha.$$

It can be shown that the mapping f is closed, and the set Φ_2 , consisting of the set of all points whose full inverse images are not bicomact, is open, everywhere dense in Y , and has the cardinality of the continuum.

2.2. Let $OXYZ$ be some rectangular coordinate system in R^3 , and let K^3 be the unit cube:

$$K^3 = E\{(x, y, z) \in R^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}.$$

Consider the sets α and β of the cube K^3 , defined in the following way:

$$\alpha = E\{(x, y, z) \in K^3 : x \in C, y = 0, z \in C\}$$

and

$$\beta = E\{(x, y, z) \in K^3 : x \in C, y = 0, z \in C\},$$

where C denotes some fixed zero-dimensional compact subset of the interval $[0, 1]$ of the cardinality of the continuum (for example, the Cantor perfect set). Define the set Φ as follows:

$$\Phi = E\{(x, y, z) \in \alpha : x \in C, y = 0, z = 0\}.$$

On the set

$$X = (\alpha \setminus \Phi) \cup \beta$$

we introduce a topology in the following way: represent β as the disjoint sum of the sets C_λ , i.e.

$$\beta \bigcup_{\lambda} C_\lambda, \quad C_\lambda \cap C_\gamma = \emptyset, \quad \text{for } \lambda \neq \gamma,$$

where each $C_\lambda = C$ has the topology induced from the interval $[0, 1]$, parallel to the axis OZ . Let $x \in \beta$; then there exists a unique $C_{\lambda(x)}$ such that $x \in C_{\lambda(x)}$. As a neighborhood of the point x we declare an arbitrary open neighborhood of it in the topology of $C_{\lambda(x)}$. If the point $x \in \alpha \setminus \Phi$, then its neighborhood Ox in X is defined by the formula

$$Ox = O'x \cup \{P(O'x) \setminus B\}, \quad O'x = Vx \cap (\alpha \setminus \Phi),$$

where Vx is a rectangular neighborhood of the point x in the natural topology of the face α , having compact closure in α ; $P(O'x)$ is the projection of $O'x$ onto β , and the set B is either empty or ...

$$B = \bigcup_{i=1}^n \Phi'_{\lambda_i},$$

where each $\Phi'_{\lambda_i} = T_i \cap C_{\lambda_i}$, and T_i is an open connected set on the interval $[0, 1]$, containing C_{λ_i} .

It can be shown that the space X in this topology is locally bicomact, σ -paracompact, weakly paracompact, nonnormal, a strict p -space, and does not admit a perfect mapping onto a space with a refining sequence of covers.

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