

# ON NECESSARY AND SUFFICIENT CONDITIONS FOR POINTS OF THE MARKOV SPECTRUM TO BELONG TO THE LAGRANGE SPECTRUM

A. A. Bershtein

1970

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.01196>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 511.7

**MATHEMATICS**

**A. A. BERSHTEIN**

# ON NECESSARY AND SUFFICIENT CONDITIONS FOR POINTS OF THE MARKOV SPECTRUM TO BELONG TO THE LAGRANGE SPECTRUM

*(Presented by Academician Yu. V. Linnik on 2 IX 1969)*

The present paper is a development of the ideas and methods of the work of G. A. Freiman <sup>(1)</sup>.

Notation:  $x$  is a sequence of natural numbers  $a_0, \dots, a_i, \dots$ ;

$$[q_1; q_2, \dots] = q_1 + \frac{1}{q_2 + \dots}; \quad \lambda_i(\mathcal{L}) = a_i + [0; a_{i+1}, \dots] + [0; a_{i-1}, \dots, a_0];$$

$$\lambda(\mathcal{L}) = \overline{\lim}_{i \rightarrow \infty} \lambda_i(\mathcal{L});$$

$\{\lambda(\mathcal{L})\}$  is the Lagrange spectrum;  $M$  is a sequence of natural numbers infinite in both directions  $\dots, a_{-1}, a_0, a_1, \dots$ ;

$$\lambda_i(M) = a_i + \delta_i + \gamma_i,$$

where

$$\delta_i = [0; a_{i+1}, a_{i+2}, \dots]; \quad \gamma_i = [0; a_{i-1}, a_{i-2}, \dots]; \quad \lambda(M) = \sup_i \lambda_i(M);$$

$\{\lambda(M)\}$  is the Markov spectrum;

$$\alpha = 4\sqrt{30}/7, \quad \beta = \sqrt{689}/8.$$

A sequence  $M_1, M_2, \dots, M_j, \dots$  will be called **convergent** to the sequence  $M$  if, for any integer  $i$ , we have

$$\lim_{j \rightarrow \infty} a_i^{(j)} = a_i, \quad \lim_{j \rightarrow \infty} \lambda(M_j) = \lambda(M).$$

A sequence  $\{M_j\}$ ,  $j = 1, 2, \dots$ , will be called **stabilizing from the right (from the left)** if there exist a natural number  $j_0$  and an integer  $i_0$  such that, for  $i > i_0$  and  $j > j_0$  (respectively  $i < i_0$ ),

$$a_i^{(j)} = a_i. \quad (1)$$

**Theorem 1.** *Let  $\lambda(M) \in (\alpha, \beta)$ . If there exists a sequence  $\{M_j\}$  converging to  $M$  and not stabilizing either from the right or from the left, then  $\lambda(M) \in \{\lambda(\mathcal{L})\}$ .*

**Proof.** Without loss of generality, we assume that  $\lambda(M) = \lambda_0(M)$ . Let, for the given  $j$ , condition (1) be satisfied for

$$\begin{aligned} -i_0 \leq i \leq i_1; \quad i_0, i_1 > 0, \\ a_{-i_0-1}^{(j)} \neq a_{-i_0-1}, \quad a_{i_1+1}^{(j)} \neq a_{i_1+1}. \end{aligned}$$

Define the sequence

$$\bar{M}_j = \{\dots, \bar{a}_{-1}, \bar{a}_0, \bar{a}_1, \dots\}$$

by the equalities:

$$\begin{aligned} \bar{a}_i = a_i, \quad -i_0 \leq i \leq i_1; \quad \bar{a}_{i_1+1} = \bar{a}_{i_1+2} = 2; \quad \bar{a}_{i_1+3} = 1; \quad \bar{a}_{i_1+4} = \bar{a}_{i_1+5} = \bar{a}_{i_1+6} = 2; \\ \bar{a}_i = \bar{a}_{i-4}, \quad i \geq i_1 + 6; \quad \bar{a}_{-i_0-1} = \bar{a}_{-i_0-2} = 2; \quad \bar{a}_{-i_0-3} = 1; \\ \bar{a}_{-i_0-4} = \bar{a}_{-i_0-5} = \bar{a}_{-i_0-6} = 2; \quad \bar{a}_i = \bar{a}_{i+4}, \quad i \leq -i_0 - 6, \end{aligned}$$

i.e.

$$\bar{M}_j = \{\dots, 2, 2, 2, 1, 2, 2, a_{-i_0}, \dots, a_{i_1}, 2, 2, 1, 2, 2, 2, \dots\}.$$

For this sequence, when  $i > i_1$ ,  $i < -i_0$ , we have

$$\lambda_i(\bar{M}_j) \leq \alpha, \quad (2)$$

since for the combinations

$$\left\{ \begin{matrix} 222 \\ i \end{matrix} \right\}, \quad \left\{ \begin{matrix} 122 \\ i \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{matrix} 212 \\ i \end{matrix} \right\}$$

we obtain the estimate  $\lambda_i \leq \alpha$ , if one takes into account that the presence in  $M$  of the combinations  $\{3\}$  and  $\{1212\}$  gives  $\lambda(M) > \beta$ .

Let now  $-i_0 \leq i \leq i_1$ . We shall show that

$$\delta_i(\bar{M}_j) \leq \max(\delta_i(M), \delta_i(M_j)), \quad (3)$$

$$\gamma_i(\bar{M}_j) \leq \max(\gamma_i(M), \gamma_i(M_j)). \quad (4)$$

We shall prove inequality (3), since (4) is proved analogously. Let  $i$  have the same parity as  $i_1 - 1$ . Suppose, for example, that  $a_{i_1+1} = 2$ ,  $a_{i_1+1}^{(j)} = 1$ . Then

$$[0; a_i, \bar{a}_{i+1}, \dots, a_{i_1}, 2, \dots] < [0; a_i, \dots, a_{i_1}, 1, \dots]$$

for any  $a_i$ ,  $i < i_1$ . Therefore

$$\delta_i(\bar{M}_j) \leq \delta_i(M_j). \quad (5)$$

Let now  $i$  have the same parity as  $i_1$ . If  $a_{i_1+2} = 1$ , then

$$[0; a_i, \dots, a_{i_1}, 2, 2, \dots] < [0; a_i, \dots, a_{i_1}, 2, 1, \dots]. \quad (6)$$

If  $a_{i_1+2} = 2$ ,  $a_{i_1+3} = 2$ , then

$$[0; a_i, \dots, a_{i_1}, 2, 2, 1, \dots] < [0; a_i, \dots, a_{i_1}, 2, 2, 2, \dots]. \quad (7)$$

If  $a_{i_1+2} = 2$ ,  $a_{i_1+3} = 1$ ,  $a_{i_1+4} = 1$ , then

$$[0; a_i, \dots, a_{i_1}, 2, 2, 1, 2, \dots] < [0; a_i, \dots, a_{i_1}, 2, 2, 1, 1, \dots]. \quad (8)$$

If  $a_{i_1+2} = 2$ ,  $a_{i_1+3} = 1$ ,  $a_{i_1+4} = a_{i_1+5} = 2$ , then  $a_{i_1+5} = 2$ , since otherwise the combination  $\{ \begin{smallmatrix} 21 & 21 \\ i & \end{smallmatrix} \}$  would occur in  $M$ . If  $a_{i_1+2} = 2$ ,  $a_{i_1+3} = 1$ ,  $a_{i_1+4} = a_{i_1+5} = 2$  and  $a_{i_1+6} = 1$ , then

$$[0; a_i, \dots, a_{i_1}, 2, 2, 1, 2, 2, 2, \dots] < [0; a_i, \dots, a_{i_1}, 2, 2, 1, 2, 2, 1, \dots]. \quad (9)$$

Each of the inequalities (6), (7), (8), (9) entails the validity of the inequality

$$\delta_i(\bar{M}_j) < \delta_i(M).$$

Since the length of the period in  $\bar{M}_j$  is equal to 4, inequality (3) is proved completely.

From (1)–(4) it follows that

$$\lim_{j \rightarrow \infty} \lambda(\bar{M}_j) = \lambda(M). \quad (10)$$

Let

$$T_j, R_j \geq \max(2i_0, 2i_1), \quad (11)$$

with  $a_{R_j}^{(j)} = 1$  and  $\bar{a}_{-T_j}^{(j)} = \bar{a}_{-T_j+1}^{(j)} = \bar{a}_{-T_j+2}^{(j)} = 2$ .

For the sequence

$$\mathcal{L} = \{a_{-T_1}^{(1)}, \dots, a_{R_1}^{(1)}, a_{-T_2}^{(2)}, \dots, a_{R_2}^{(2)}, \dots\}$$

in view of (10) and (11),

$$\lambda(\mathcal{L}) = \lambda(M).$$

A direct consequence of Theorem 1 is

**Theorem 2.** Let  $\lambda(M) \in (\alpha, \beta)$ , and suppose there exists an infinite set of sequences  $M'$  for which  $\lambda(M') = \lambda(M)$ . Then  $\lambda(M) \in \{\lambda(\mathcal{L})\}$ .

We say that for some  $\varepsilon > 0$  the sequence  $M$  has the  $\varepsilon$ -property on the right (on the left) if there exists an integer  $i_0$  such that for any sequence  $M' \neq M$  for which  $a_i = a'_i$  for  $i < i_0$  (respectively  $i > i_0$ ), we have  $\lambda(M') > \lambda(M) + \varepsilon$ . If  $\lim_{i \rightarrow +\infty} \lambda_i(M) < \lambda(M)$ , then the sequence is called indefinite.

**Theorem 3.** Let  $\lambda(M_1) \in (\alpha, \beta)$ . In order that  $\lambda(M_1) \in \overline{\{\lambda(\mathcal{L})\}}$ , the following conditions are necessary and sufficient:

1. There exists only a finite number of sequences  $\overline{M_s}$ ,  $1 \leq s \leq k$ , such that

$$\lambda(\overline{M_1}) = \lambda(\overline{M_2}) = \dots = \lambda(\overline{M_k}).$$

2. The sequences  $M_s$  are nonperiodic.
3. The sequences  $M_s$  are periodic on the right\*, i.e., there exist an integer  $i_0$  and a natural number  $p$  such that, for  $i > i_0$ ,  $a_i = a_{i+p}$ .
4. Each of the sequences  $M_s$ , periodic in both directions, with period equal to the period of the sequence  $M_s$ , has the  $\varepsilon$ -property on the right, where

$$\varepsilon = \lambda(\overline{M_s}) - \lim_{i \rightarrow \infty} \lambda_i(M_s).$$

**Proof.** The necessity of condition 1 follows from Theorem 2. Since the sequence  $M_1$  is nonperiodic (see (2)), there exist an integer  $i_1$  and  $\varepsilon_1 > 0$  such that  $\lambda_i(M_1) < \lambda(M_1) - \varepsilon_1$  for  $i > i_1$ . One can choose an  $s = s(\varepsilon_1)$  such that if in  $M_1$  and  $M'$  we have  $a_i = a'_i$  for  $i \leq j + s$  or  $i \geq j - s$ , then

$$\lambda_j(M') < \lambda_j(M) + \frac{\varepsilon_1}{2}. \quad (12)$$

There will be natural numbers  $b_1, b_2, \dots, b_{2s}$  and sequences  $i_1, i_2, \dots, i_r, \dots$  such that  $a_{i_r+t} = b_{t+1}$ ,  $t = 0, 1, \dots, 2s-1$ ,  $r = 1, 2, \dots$ . Form the sequences

$$M^{(r)} = \{a_i^{(r)} = a_i, \ i \leq i_r, \ a_i^{(r)} = a_{i+i_1-i_r}, \ i > i_r\}. \quad (13)$$

For them, in view of (12) and (13),

$$\lambda_i(M^{(r)}) < \lambda_i(M_1) + \frac{\varepsilon_1}{2}, \quad i \leq i_r + s.$$

In view of (12),

$$\lambda_i(M^{(r)}) < \lambda(M_1), \quad i > i_r + s.$$

Analogous arguments can also be carried out for small  $i$ . If we assume that  $M_1$  is nonperiodic both on the right and on the left, then it would now follow from Theorem 1 that  $\lambda(M_1) \in \overline{\{\lambda(\mathcal{L})\}}$ .

Finally, condition 4 is also satisfied, since otherwise it would follow from Theorem 1 that  $\lambda(M_1) \in \overline{\{\lambda(\mathcal{L})\}}$ .

Conditions 1-4 are also sufficient in order that  $\lambda(M_1) \in \overline{\{\lambda(\mathcal{L})\}}$ . Indeed, if  $\lambda(M_1) = \lambda(\mathcal{L})$  for some  $\mathcal{L}$ , then in  $\mathcal{L}$  there would occur arbitrarily long segments of the sequences  $M^{(r)}$ , and this contradicts conditions 1-4.

All the theorems of the present paper are formulated only for  $\lambda(M) \in (\alpha, \beta)$ . The set for which Theorems 1-3 are valid can be substantially enlarged.

Central Economics and Mathematics Institute  
Academy of Sciences of the USSR  
Moscow

Received  
14 VI 1969

## CITED LITERATURE

1. G. A. Freiman, *Mathematical Notes*, **3**, No. 2, 195 (1968).
2. P. G. Kogonia, *Proceedings of the Tbilisi Mathematical Institute*, **29**, 15 (1963).

\* The sequence  $M$  can always be replaced by  $M'$ , for which  $b_i = a_{-i}$ , and we shall not distinguish between two such sequences.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*