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Abstract

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PHYSICS

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CLUMPS OF DIAMAGNETIC PLASMA IN THE FIELD OF A MAGNETIC DIPOLE

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In the present note we consider the problem of the equilibrium configuration of a diamagnetic plasma in the magnetic field of a two-dimensional dipole. Such a problem may be of interest in the study of certain problems of geophysics, as well as in considering phenomena in the solar chromosphere.

Let us consider the field of a plane magnetic dipole with complex potential

$$w = -iM/2\pi z = \varphi + i\psi. \quad (1)$$

Here M is the dipole moment, which remains constant, $z = x + iy$. The lines of force will be the lines $\psi = -M/4\pi a = \text{const}$, where a are the radii of circles with centers on the x axis and tangent to the y axis. Thus, ψ is the component of the vector potential normal to the xy plane. The magnetic field $\mathbf{B} = B_x + iB_y$ is determined from the equality

$$dw/dz = B_x - B_y. \quad (2)$$

Suppose that in such a field, along the line of force ψ_0 , there is a plasma with isotropic pressure p_0 , which is maintained constant. Then the configuration assumed by the plasma in equilibrium must satisfy the boundary conditions:

$$B_0^2/8\pi = p_0 \quad (\mathbf{B}_0 \mathbf{n}) = 0 \quad (3)$$

($\mathbf{n}$ is the outward normal to the plasma boundary).

To solve the problem posed, we introduce the dimensionless variable

$$\xi = (iB^*/B_0)^{1/2}. \quad (4)$$

In the ξ plane the required configuration is mapped onto an arc of the unit circle (Fig. 1), and the problem reduces to finding the complex po—

Figure 1

Figure 1: Figure 1

Fig. 1. *a*—pattern of the magnetic field in the presence of plasma (hatched region) in the z plane; z_0 —branch point of the line of current; x_1, x_2 —boundary points of the region on the axis of abscissas. *b*—in the ξ plane the region occupied by the plasma is mapped onto an arc of the unit circle (the imaginary y axis is mapped onto the real axis in the ξ plane, and the real x axis—onto the imaginary semi-axis).

potential of the flow past the arc near the flat boundary, under certain conditions with respect to circulation.

The velocity v_0 at infinity is determined as follows:

$$\mathbf{v}_0 = \left. \frac{dw}{d\xi} \right|_{\xi=\infty} = v_0 e^{i\vartheta}, \quad v_0 = (MB_0/2\pi)^{1/2}, \quad \vartheta = \pi. \quad (5)$$

According to Chaplygin [1], the required complex potential is determined by the formula

$$\frac{1}{v_0} \frac{dw}{d\xi} = \frac{1}{2} \left(e^{-i\vartheta} - \frac{1}{\xi^2} e^{i\vartheta} \right) + \left(\frac{1}{2} e^{-i\vartheta} + \frac{1}{2} \frac{d^2 e^{i\vartheta}}{\xi^4} + \frac{G}{\xi} + \frac{d^2 G^*}{\xi^3} + \frac{d\gamma}{\xi^2} \right) \frac{1}{R}; \quad (6)$$

$$R = \left[\prod (1 - a_i/\xi) \right]^{1/2};$$

a_i are the end points of the arcs; $d^4 = \prod a_i$ is a complex number with modulus 1; G, G^* are conjugate complex numbers; γ is a real coefficient. We note that R changes sign on different sides of the arcs.

From the condition that the zero point in the ξ -plane is an ordinary point for w , while the circulation around infinity must be zero, we have

$$G = G^* = 0. \quad (7)$$

By the condition of the problem, the branch points of the streamlines coincide with the ends of the arc, in accordance with which we put

$$\gamma = \cos 2\alpha_0.$$

Here α_0 is the angle between the radius vector of the end of the arc and the abscissa axis. Thus, the complex potential of the field in the presence of plasma as a function of ξ is expressed by the formula

$$\frac{1}{v_0} \frac{dw}{d\xi} = \frac{1}{2} \left(\frac{1}{\xi^2} - 1 - \sqrt{1 + \frac{1}{\xi^4} - \frac{2 \cos 2\alpha_0}{\xi^2}} \right). \quad (8)$$

The shape of the boundary of the region in the xy -plane occupied by the plasma is determined from (2), (4), and (8):

$$\int dz = \int \frac{dw}{B^*} = \frac{iv_0}{2B_0} \int \left(\frac{1}{\xi^2} - 1 - \sqrt{1 + \frac{1}{\xi^4} - \frac{2 \cos 2\alpha_0}{\xi^2}} \right) \frac{1}{\xi^2} d\xi. \quad (9)$$

By integrating in the corresponding limits, points belonging to the plasma boundary are determined. In view of the symmetry of the problem, only the first quadrant of the xy -plane is considered. The calculations are rather cumbersome, but not difficult [5, 6]; therefore we give here only the results:

$$\begin{aligned} z - x_2 = & \frac{v_0}{B_0} \left\{ \frac{2}{3} (\sin^3 \alpha - 1) - \left[\cos \alpha_0 \left(\sin^2 \alpha_0 - \frac{1}{3} \cos^2 \alpha_0 + \gamma \right) \right. \right. \\ & \left. \left. + (\cos^2 \alpha_0 - \cos^2 \alpha)^{1/2} \left(\frac{2 \cos^2 \alpha + 4 \cos^2 \alpha_0}{3} - 1 - \gamma \right) \right] \right\} \\ & + i \frac{v_0}{3B_0} \left\{ \cos \alpha (2 \sin^2 \alpha + 1) + \sin 2\alpha (\sin^2 \alpha - \sin^2 \alpha_0)^{1/2} \right. \\ & \left. + \frac{1}{2} [(3 - 4 \sin^2 \alpha_0 - 3\gamma) F(\psi, \cos \alpha_0) - (4 - 8 \sin^2 \alpha_0 - 6\gamma) E(\psi, \cos \alpha_0)] \right\}; \quad (10A) \end{aligned}$$

$$\begin{aligned} \tilde{z} - z_0 = & \frac{v_0}{B_0} \left\{ \frac{2}{3} (\sin^2 \alpha - \sin^3 \alpha_0) + (\cos^2 \alpha_0 - \cos^2 \alpha)^{1/2} \left(\frac{2 \cos^2 \alpha + 4 \cos^2 \alpha_0}{3} - 1 - \gamma \right) \right\} \\ & + i \frac{v_0}{3B_0} \left\{ \cos \alpha (2 \sin^2 \alpha + 1) - \cos \alpha_0 (2 \sin^2 \alpha_0 + 1) - \sin 2\alpha (\sin^2 \alpha - \sin^2 \alpha_0)^{1/2} \right. \\ & - \frac{1}{2} \left[(3 - 4 \sin^2 \alpha_0 - 3\gamma) \int_{\cos \alpha / \cos \alpha_0}^0 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} \right. \\ & \left. \left. - (4 - 8 \sin^2 \alpha_0 - 5\gamma) \int_{\cos \alpha / \cos \alpha_0}^1 \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} \right] \right\}. \quad (10B) \end{aligned}$$

Here and below the following notation is used: z denotes points belonging to the outer part of the boundary of the plasma region, which bounds the inner part at the branch point z_0 ; $\tilde{z}$ denotes points belonging to the inner (facing the origin of coordinates) part of the boundary of the region; x_1, x_2 are the boundary points of the region on the abscissa axis (see Fig. 1); $F(\psi, k)$, $E(\psi, k)$ are

elliptic integrals of the corresponding arguments, of the first and second kind, respectively; $\tilde{k}^2 = \cos^2 \alpha_0$; $\psi = \arcsin(\cos \alpha / \cos \alpha_0)$; on the arc the notation $\xi = e^{i\alpha}$ is adopted.

Putting $\alpha = \alpha_0$ in (10A), and $\alpha = \pi/2$ in (10B), we obtain

$$z_0 - x_2 = \frac{v_0}{B_0} \left\{ \frac{2}{3} (\sin^3 \alpha_0 - 1) - \cos \alpha_0 \left(\sin^2 \alpha_0 - \frac{1}{3} \cos^2 \alpha_0 + \gamma \right) \right\} + i \frac{v_0}{3B_0} \left\{ \cos \alpha_0 (2 \sin^2 \alpha_0 + 1) + \frac{1}{2} \left[(3 - 4 \right. \right.$$

$$\left. x_1 - z_0 = \frac{v_0}{B_0} \left\{ \frac{2}{3} (1 - \sin^3 \alpha_0) - \cos \alpha_0 \left(\sin^2 \alpha_0 - \frac{1}{3} \cos^2 \alpha_0 + \gamma \right) \right\} + i \frac{v_0}{3B_0} \left\{ -\cos \alpha_0 (2 \sin^2 \alpha_0 + 1) - \right. \right. \quad (11)$$

The width of the plasma region along the abscissa axis is characterized by the quantity

$$x_2 - x_1 = \frac{2v_0}{B_0} \cos \alpha_0 \left(\frac{1}{3} \cos^2 \alpha_0 - \sin^2 \alpha_0 - \gamma \right), \quad (12)$$

where x_1 is determined by integration of (9) along the imaginary axis from $i\infty$ to i and has the form:

$$x_1 = \frac{v_0}{B_0} \left\{ \frac{1}{3} \left[2 - \cos \alpha_0 - 2 (1 - 2 \sin^2 \alpha_0) (2 \sin \alpha_0 - \cos \alpha_0) \right] + C_1 F \left(\frac{\pi}{2}, k \right) + C_2 E \left(\frac{\pi}{2}, k \right) - \gamma \cos \alpha_0 \right\};$$

$$C_1 = -\frac{1}{1 + \sin \alpha_0} \left[\frac{\gamma \sin^2 \alpha_0}{2} + \frac{\cos^2 \alpha_0 (1 + 2 \sin^2 \alpha_0)}{3} \right]; \quad (13)$$

$$C_2 = (1 + \sin \alpha_0) \left(\frac{\gamma}{2} + \frac{1 - 2 \sin^2 \alpha_0}{3} \right); \quad k^2 = \frac{4 \sin \alpha_0}{(1 + \sin \alpha_0)^2}.$$

In all the expressions written above there appears the parameter α_0 , which is related in a definite way to the plasma pressure. Defining the vector potential ψ_0 in terms of α by the formula

$$\psi_0 = \int_{x_2}^{\infty} B_y dx = \text{Im} \left\{ \frac{v_0}{2} \int_0^i \left(\frac{1}{\xi^2} - 1 - \sqrt{1 + \frac{1}{\xi^4} - \frac{2 \cos 2\alpha_0}{\xi^2}} \right) d\xi \right\} \quad (14)$$

and transforming the elliptic integrals obtained (see ^{5,6}), we obtain the equation

$$t = \frac{\psi_0}{v_0} = \frac{1}{2} \left[\frac{\cos^2 \alpha_0}{1 + \sin \alpha_0} F\left(\frac{\pi}{2}, k\right) - (1 + \sin \alpha_0) E\left(\frac{\pi}{2}, k\right) \right], \quad (15)$$

where k is determined as in (13), and t , from (3) and (5), has the form

$$t = \frac{\psi_0}{\sqrt{M(2p_0/\pi)^{1/2}}} \sim \frac{\psi_0}{p_0^{1/4}}. \quad (16)$$

From (15) it is not difficult to see that, as α_0 varies from 0 to $\pi/2$, the absolute value of t takes values in the interval $(0, 1)$. This means that only for such values of $|t|$ is retention of the plasma in the given trap possible. Let us examine in more detail two limiting cases.

1. **High pressures.** $|t| \rightarrow 0$. Expanding (11A)–(13) in powers of a_0 and discarding terms of orders higher than 1 in a_0 , we arrive at the following quantities characterizing the shape of the plasma region:

$$x_1 \approx 0.78v_0/B_0, \quad \text{Re } z_0 \approx x_1, \quad \text{Im } z_0 \approx 0.67v_0/B_0,$$

$$x_2 - x_1 \approx 1.33v_0/B_0. \quad (17)$$

As can be seen, at high pressures the plasma compresses the inner field lines of the dipole toward the center and at the same time stretches the outer field lines, crowding them. In this process the configuration remains closed. It is obvious that, on moving away from the outer boundary of the plasma, the field decreases much more rapidly than the field of the dipole. For $\psi_0 = 0$ the result coincides with the previously considered problem of the compression of the field of a linear dipole by a plasma of constant pressure ⁽²⁻⁴⁾.

2. **Very low pressures.** $|t| \rightarrow 1$. As $a_0 \rightarrow \pi/2$, all integrals taken over the arc tend to zero, so that all characteristic distances (17) decrease, while

$$x_1 \sim \ln(16/\beta^2) \rightarrow \infty.$$

Here $\beta = \pi/2 - a_0 \rightarrow 0$. This result means that if the plasma pressure is not sufficiently large in comparison with the local magnetic pressure, then it cannot be confined in the given region and is pushed out by the magnetic field.

In conclusion, we note that by the proposed method it appears possible to solve a number of other problems on configurations of plasma clumps in inhomogeneous magnetic fields.

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