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Abstract

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MATHEMATICS

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ON THE STRUCTURE OF THE SET OF POSITIVE DEVIATIONS OF MEROMORPHIC FUNCTIONS

(Presented by Academician M. A. Lavrentiev on 29 IV 1969)

§ 1. In order to study deeper asymptotic properties of functions meromorphic in $|z| < R$, the papers ⁽¹⁻³⁾ introduced the notion of the magnitude of deviation of a meromorphic function $f(z)$

$$\beta(a, f) = \lim_{r \rightarrow R} \frac{\ln^+ M(r, a, f)}{T(r, f)},$$

where

$$M(r, a, f) = \max_{|z|=r} \frac{1}{|f(z) - a|}, \quad a \neq \infty,$$

$$M(r, \infty, f) = \max_{|z|=r} |f(z)|,$$

and $T(r, f)$ is the Nevanlinna characteristic of the function $f(z)$ meromorphic in $|z| < R$.

The results obtained on the properties of the quantities of deviations of $f(z)$ show that the distribution of values of meromorphic functions can be studied using not only the notion of the defect at the point a in the sense of Nevanlinna (see ⁽⁴⁾)

$$\delta(a, f) = \lim_{r \rightarrow R} \frac{m(r, a, f)}{T(r, f)}$$

and the notion of the defect of $f(z)$ at the point a in the sense of Valiron

$$\Delta(a, f) = \overline{\lim}_{r \rightarrow R} \frac{m(r, a, f)}{T(r, f)},$$

but also the notion of the magnitude of deviation of $f(z)$ at the point a .

Let, for a function $f(z)$ meromorphic in $|z| < R$ and of lower order* λ (see ⁽⁴⁾, p. 267),

$$D_{R,\lambda}(f) = \{a : \delta(a, f) > 0\},$$

$$\Omega_{R,\lambda}(f) = \{a : \beta(a, f) > 0\},$$

$$V_{R,\lambda}(f) = \{a : \Delta(a, f) > 0\}.$$

The set $D_{R,\lambda}(f)$ is called the set of defective values of $f(z)$ in the sense of Nevanlinna; the set $\Omega_{R,\lambda}(f)$ is naturally called the set of positive deviations of $f(z)$, and the set $V_{R,\lambda}(f)$ is called the set of defective values of $f(z)$ in the sense of Valiron.

Fundamental investigations of the structure of the sets $D_{R,\lambda}(f)$ and $V_{R,\lambda}(f)$ were carried out in ⁽⁴⁻⁸⁾.

* Recall that the lower order of a meromorphic function in $|z| < R$ is called

$$\lambda = \lim_{r \rightarrow R} \frac{\ln T(r, f)}{\ln \frac{1}{R-r}}, \quad R < \infty, \quad \lambda = \lim_{r \rightarrow \infty} \frac{\ln T(r, f)}{\ln r}, \quad R = \infty.$$

This work is devoted to the study of the structure of the set $\Omega_{R,\lambda}(f)$. The results obtained show that, despite possible substantial differences between the sets $\Omega_{R,\lambda}(f)$, $D_{R,\lambda}(f)$, and $V_{R,\lambda}(f)$, nevertheless in most cases the set $\Omega_{R,\lambda}(f)$, just like the sets $D_{R,\lambda}(f)$ and $V_{R,\lambda}(f)$, is an exceptional set for $f(z)$.

§ 2. Main results of the work.

Theorem 1 (see ^(1,2)). For $\lambda < \infty$, the set $\Omega_{\infty,\lambda}(f)$ is at most countable, and

$$D_{\infty,\lambda}(f) \subseteq \Omega_{\infty,\lambda}(f) \subseteq V_{\infty,\lambda}(f).$$

The following three theorems characterize the differences between the deviations of meromorphic functions $\beta(a, f)$ and the values of their Nevanlinna defects $\delta(a, f)$ and Valiron defects $\Delta(a, f)$.

Theorem 2. The sets $\Omega_{\infty,\infty}(f)$ and $\Omega_{\infty,\infty}(f) \setminus V_{\infty,\infty}(f)$ may have the cardinality of the continuum.

Theorem 3. *There exists a set C of the cardinality of the continuum and an entire function $G(z)$ of infinite lower order such that $\beta(a, G) = \infty$ for every $a \in C$.*

Theorem 4. *For every ρ , $0 \leq \rho \leq \infty$, there exists a set C of the cardinality of the continuum and a function $g_\rho(z)$, meromorphic for $|z| < 1$, of order ρ , such that $\beta(a, g_\rho) = \infty$ for every $a \in C$.*

Corollary 1. *For any $\rho \geq 0$, the set $\Omega_{1,\rho}(f)$ may have the cardinality of the continuum.*

Let us note that for $\rho = 0$ Theorem 4 follows from the consideration of functions of bounded type (see ⁽⁴⁾, p. 210).

Next, for a function $f(z)$ meromorphic in $|z| < R$, for $0 < \alpha \leq 1$, put

$$\beta_\alpha(a, f) = \lim_{r \rightarrow R} \frac{\ln^+ M(r, a, f)}{T^\alpha(r, f)}$$

$$\Omega_{R,\lambda}^{(\alpha)}(f) = \{a : \beta_\alpha(a, f) > 0\}.$$

The following assertions are certain analogues of the well-known Ahlfors–Nevanlinna theorem (see ⁽⁵⁾, p. 17; ⁽⁴⁾, p. 281) on the structure of the set $V_{R,\lambda}(f)$. In what follows, by E we denote a bounded set in the a -plane having positive capacity $C(E)$, and by $\mu(a)$ the distribution of unit mass on E solving Robin's problem for E (see ⁽⁴⁾, pp. 123, 135).

Theorem 5. *For a function $f(z)$ meromorphic at $z \neq \infty$, the following assertions are valid:*

1) *For every $\alpha > 1/2$*

$$\int_E \beta_\alpha(a, f) d\mu(a) = 0;$$

2) *if $E \subset \{|a| \leq 1\}$, $C(E)$ is the capacity of the set E , and λ is the lower order of $f(z)$ ($f(0) = 1$), then*

$$\int_E \beta_{1/2}(a, f) d\mu(a) \leq K_1(1 + \lambda^2) \left[1 + \ln^+ \frac{1}{C(E)} \right].$$

Corollary 2. *For every $\alpha > 1/2$, the set $\Omega_{\infty,\lambda}^{(\alpha)}(f)$ has inner capacity zero ($\Omega_{\infty,\lambda}^{(1)}(f) = \Omega_{\infty,\lambda}(f)$, $\lambda \geq 0$).*

Theorem 6. *If $f(z)$ is meromorphic for $|z| < 1$ and has lower order λ , then for every $\alpha > 1/2 + 3/\lambda$ ($\lambda > 0$)*

$$\int_E \beta_\alpha(a, f) d\mu(a) = 0.$$

Corollary 3. If $\alpha > 1/2 + 3/\lambda$, then the set $\Omega_{1,\lambda}^{(\alpha)}(f)$ has inner capacity zero.

§ 3. Theorem 1 was proved in works ^(1, 2).

Theorems 2, 3, and 4 are proved by means of a modification of known constructions ⁽⁶⁻¹⁰⁾.

Theorems 5 and 6 are proved by means of a modification of a method previously used by the author ^(1, 2).

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