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Abstract

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MATHEMATICS

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ON PERTURBATION OF THE CONTINUOUS SPECTRUM

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This paper gives a generalization of the results obtained in ⁽¹⁾ to the case of infinite-dimensional perturbations.

1. As the unperturbed operator we still take the operator S of multiplication by the independent variable in the Hilbert space $H = L_2(-\infty, \infty)$, with domain $D(S)$ consisting of those $f(x) \in H$ for which also $xf(x) \in H$, and $Sf(x) = xf(x)$. To describe the perturbed operator we shall need several definitions.

Let ε be some positive number. By $\Phi = \Phi(\varepsilon)$ we denote the set of those $\varphi \in H$ for which there exists a continuation $\varphi(z)$ of the function $\varphi(x)$, holomorphic for $|\operatorname{Im} z| < \varepsilon$ and satisfying the condition

$$\sup_{|y| \leq \varepsilon_1} \int_{-\infty}^{\infty} |\varphi(x + iy)|^2 dx < \infty, \quad \varepsilon_1 < \varepsilon. \quad (1)$$

Functions $\varphi \in \Phi$ will be called **completely regular** elements of H .

Let G be some (auxiliary abstract) Hilbert space and A a linear continuous operator: $H = D(A) \rightarrow G$. If the range $R(A^*)$ of the adjoint operator $A^* : G = D(A^*) \rightarrow H$ consists of completely regular elements, i.e.

$$R(A^*) \subset \Phi, \quad (2)$$

then, as is not hard to prove, there exists a vector-valued function $\alpha(z)$ with values in G , holomorphic for $|\operatorname{Im} z| < \varepsilon$, such that

$$Af = \int_{-\infty}^{\infty} f(x)\alpha(x) dx, \quad f \in H, \quad (3)$$

where the integral exists in the sense of weak convergence in G . If one assumes that

$$\int_{-\infty}^{\infty} \|\alpha(x)\|_G^2 dx < \infty, \quad (4)$$

then A will be a completely continuous operator $H \rightarrow G$. Such an operator (i.e. an operator satisfying conditions (2) and (4)) we call **completely regular**.

Definition 1. An operator $V : H = D(V) \rightarrow H$ is called a **completely regular perturbation** if it admits a representation of the form $V = A^*B$, where A and B are completely regular operators $H \rightarrow G$.

In this paper the operator $T \stackrel{\text{df}}{=} S + V$, $D(T) \stackrel{\text{df}}{=} D(S)$, is studied, where V is a completely regular perturbation*. Since the norm of V is not assumed to be small, the operator T , generally speaking, is not similar to S . In addition to eigenvalues of the usual type, the operator T may have spectral singularities (see Sec. 2), which, in particular, may be eigenvalues with nonzero index. The eigenfunctions of the continuous spectrum of the operator T turn out to be δ -functions perturbed by boundary values of certain Cauchy-type integrals (see Sec. 3). The Parseval equality corresponding to the operator T is written by means of regularization of a certain integral that diverges near the spectral singularities.

2. It is convenient to describe the spectral properties of the operator T in terms of the operator function $K(\zeta) : G \rightarrow G$

$$K(\zeta) \stackrel{\text{df}}{=} 1 + BS_\zeta A^*, \quad \text{Im } \zeta \neq 0, \quad (5)$$

where $S_\zeta \stackrel{\text{df}}{=} (S - \zeta)^{-1}$. For example, it is not hard to see that a nonreal λ is an eigenvalue of the operator T if and only if the operator $K(\lambda)$ is not invertible in G . If $\text{Im } \zeta \neq 0$ and $K(\zeta)^{-1}$ exists, then ζ belongs to the resolvent set of the operator T , and

$$T_\zeta \stackrel{\text{df}}{=} (T - \zeta)^{-1} = S_\zeta - S_\zeta A^* K(\zeta)^{-1} B S_\zeta. \quad (6)$$

The set of eigenvalues of the operator T is finite, and to each eigenvalue there corresponds a finite-dimensional root subspace.

Formula (6) makes it possible to conclude that, for any $\varphi, \psi \in \Phi$, the function $(T_\zeta \varphi, \psi)$ (of the variable ζ) has continuations $(T_\zeta \varphi, \psi)_+$ and $(T_\zeta \varphi, \psi)_-$ from the half-planes respectively $\text{Im } \zeta > 0$ and $\text{Im } \zeta < 0$, analytic in the half-planes respectively $\text{Im } \zeta > -\varepsilon$ and $\text{Im } \zeta < +\varepsilon$, and having no singularities other than poles.

Definition 2. The real poles of the functions $(T_\zeta \varphi, \psi)_+$ and $(T_\zeta \varphi, \psi)_-$ are called **spectral singularities** of the operator T .

The set of spectral singularities of the operator T is also finite and coincides with the set of spectral singularities of the operator T^* . In order that a real σ be an eigenvalue of the operator T or T^* , it is necessary (but not sufficient) that σ be a spectral singularity. In this case the dimension of the root subspace corresponding to T^* , generally speaking, is not equal to the dimension of the root subspace corresponding to T , so that, generally speaking, σ has nonzero index. The spectral singularities coincide with the real poles of the operator-valued functions $K_+(\zeta)^{-1}$ and $K_-(\zeta)^{-1}$, where $K_+(\zeta)$ and $K_-(\zeta)$ are the analytic continuations of $K(\zeta)$ from the upper and lower half-planes, respectively.

To each nonreal eigenvalue λ of the operator T we assign the eigenprojection

$$P_\lambda \stackrel{\text{df}}{=} -\frac{1}{2\pi i} \oint T_\zeta d\zeta, \quad (7)$$

where inside the contour of integration there are no singularities of the resolvent T_ζ other than λ . To each spectral singularity σ we assign two “eigenprojections” $P_{\sigma+}$ and $P_{\sigma-}$,

$$(P_{\sigma\pm}\varphi, \psi) = -\frac{1}{2\pi i} \oint (T_\zeta\varphi, \psi)_\pm d\zeta, \quad \varphi, \psi \in \Phi, \quad (8)$$

* Some of the results formulated below are also valid under more general assumptions on the perturbation V .

with an analogous condition on the contour of integration. We note that the “projectors” $P_{\sigma\pm}$ act from Φ into the set of semilinear functionals defined on Φ . All the operators P_λ , $P_{\sigma+}$, and $P_{\sigma-}$ are finite-dimensional. It is useful to apply formula (6) in order to express P_λ , $P_{\sigma+}$, and $P_{\sigma-}$ in terms of the residues of $K(\zeta)$, $K_+(\zeta)$, and $K_-(\zeta)$ at $\zeta = \lambda$ and $\zeta = \sigma$, respectively.

3. If the real ξ is not a spectral singularity, then ξ belongs to the continuous spectrum of the operator T . In order to describe the jump of the resolvent T_ξ on the continuous spectrum, we introduce the operators:

$$\mathfrak{A}\varphi(\xi) \stackrel{\text{df}}{=} \varphi(\xi) - B^*K_+(\bar{\xi})^{*-1}AS_\xi\varphi_-(\xi), \quad \varphi \in \Phi, \quad |\text{Im } \xi| < \varepsilon, \quad (9)$$

$$\mathfrak{B}\varphi(\xi) \stackrel{\text{df}}{=} \varphi(\xi) - A^*K_-(\xi)^{-1}BS_\xi\varphi_-(\xi);$$

here $AS_\xi\varphi_-$ and $BS_\xi\varphi_-$ are continuations from the lower half-plane of the functions $AS_\xi\varphi$ and $BS_\xi\varphi$, which exist and are holomorphic (in the metric of G) if $\varphi \in \Phi$. It turns out that, for arbitrary $\varphi, \psi \in \Phi$ and for all ξ from some neighborhood of the real axis, with the exception of spectral singularities,

$$(T_\xi\varphi, \psi)_+ - (T_\xi\varphi, \psi)_- = 2\pi i \mathfrak{B}\varphi(\xi)\overline{\mathfrak{A}\psi(\xi)}. \quad (10)$$

From the “jump” formula (10) it is easy to derive that

$$\mathfrak{A}T^*\varphi(\xi) = \xi\mathfrak{A}\varphi(\xi), \quad \mathfrak{B}T\varphi(\xi) = \xi\mathfrak{B}\varphi(\xi) \quad (11)$$

for all real ξ distinct from the spectral singularities, which gives grounds for regarding the operators $\mathfrak{A}$ and $\mathfrak{B}$ as, respectively, the T^* - and T -Fourier transforms on the continuous spectrum. To describe the continuity properties of these operators, put*

$$\kappa_{\pm}(\zeta) \stackrel{\text{df}}{=} \prod_{\sigma} \left(\frac{\zeta - \sigma}{\zeta - i\varepsilon} \right)^{n_{\sigma\pm}}, \quad (12)$$

where the product sign extends over all spectral singularities σ , and $n_{\sigma\pm}$ denotes the order of σ as a pole of $K_{\pm}(\zeta)^{-1}$. Then the operators

$$\mathfrak{A}_0\varphi(\xi) \stackrel{\text{df}}{=} \overline{\kappa_+(\xi)} \mathfrak{A}\varphi(\xi), \quad \mathfrak{B}_0\varphi(\xi) \stackrel{\text{df}}{=} \kappa_-(\xi) \mathfrak{B}\varphi(\xi) \quad (13)$$

admit extension from Φ to continuous operators $H \rightarrow H$. Moreover, if $\mathcal{D}$ denotes the differentiation operator in H , then from $f \in D(\mathcal{D}^j)$ it follows that $\mathfrak{A}_0f, \mathfrak{B}_0f \in D(\mathcal{D}^j)$ and

$$\|\mathcal{D}^j\mathfrak{A}_0f\|, \|\mathcal{D}^j\mathfrak{B}_0f\| \leq C_j \left(\sum_{j'=1}^j \|\mathcal{D}^{j'}f\|^2 \right)^{1/2}, \quad j = 0, 1, 2, \dots, \quad (14)$$

where C_j are certain constants.

The estimates (14) (for $j = 1$) allow one to speak of the values of the functions $\mathfrak{A}_0f(\xi)$, $\mathfrak{B}_0f(\xi)$ for $f \in D(\mathcal{D})$ for all real ξ . Denoting them respectively by (f, a_{ξ}) and (f, b_{ξ}) , and taking (11) into account, we may regard the functionals a_{ξ} and b_{ξ} as (generalized) eigenfunctions of the continuous spectrum of the operators T and T^* , respectively.

4. In conclusion we describe one variant of Parseval's equality for the operator T . Put

$$\int_+ \frac{f(x) dx}{(x - \sigma)^{\nu}} \stackrel{\text{df}}{=} \lim_{\tau \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x) dx}{[x - (\sigma + i\tau)]^{\nu}} \quad (15)$$

* Since the operator $K_{\pm}(\zeta) - 1$ is nuclear, there exists a determinant of the operator $K_{\pm}(\zeta)$. Everything that follows remains valid if relation (12) is replaced by the relation

$$\kappa_{\pm}(\zeta) \stackrel{\text{df}}{=} \det K_{\pm}(\zeta).$$

provided that the limit on the right exists. Let $n = \max(n_{\sigma+} + n_{\sigma-})$, where the maximum is taken over all spectral singularities σ , and the numbers $n_{\sigma\pm}$ are the same as in (12). Then, for all $f, g \in D(\mathcal{D}^n)$,

$$\int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = \int_{+} (f, b_{\xi}) \overline{(g, a_{\xi})} \frac{d\xi}{\varkappa_{+}(\xi) \varkappa_{-}(\xi)} + \sum_{\sigma} (P_{\sigma} f, g) + \sum_{\lambda} (P_{\lambda} f, g). \quad (16)$$

Here a_{ξ} and b_{ξ} are eigenfunctions of the continuous spectrum of T and T^* (see item 3); P_{λ} and P_{σ} are the eigenprojections of eigenvalues and spectral singularities (see (7) and (8)); let us clarify that P_{σ} is extended by continuity, in the norm $(\|f\|^2 + \dots + \|f^{(n)}\|^2)^{1/2}$ from Φ to $D(\mathcal{D}^n)$, while $\sum_{\lambda}$ ranges over all nonreal eigenvalues λ , and $\sum_{\sigma}$ over all spectral singularities σ . As is seen from (12), each spectral singularity σ is a pole of order $n_{\sigma+} + n_{\sigma-}$ of the “spectral” density function $1/\varkappa_{+}(\sigma) \varkappa_{-}(\sigma)$. In this connection, a regularized integral $\int_{+}$ arises on the right-hand side of (16). We understand this integral, roughly speaking, in the sense of (15)*. The existence of such a regularization is ensured by the estimates (14).

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References

1. V. E. Lyantse, DAN, **182**, No. 5 (1968).

* The corresponding partition of unity is used, making it possible to find the limit (15) for each spectral singularity separately.

Note: Figure translations are in progress. See original paper for figures.

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