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A. I. Shalyt

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Abstract

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MATHEMATICS

A. I. SHALYT

SOME RESULTS ON SEQUENTIAL ESTIMATION OF A SHIFT PARAMETER

(Presented by Academician Yu. V. Linnik on 26 III 1969)

Let $(x_1, x_2, \dots, x_n)$ be independent identically distributed random variables with distribution function (d.f.) $F(x - \theta)$, depending on the shift parameter $\theta \in R^1$. The problem of estimating the parameter θ from a sample $(x_1, x_2, \dots, x_n)$ of fixed size, after the fundamental work of Pitman ⁽¹⁾, has been the subject of papers ⁽²⁻⁵⁾, which substantially advanced the theory of estimation of a shift parameter. However, there are still many unsolved problems here, one of which is considered by us in § 1.

In our view, the problem of sequential estimation of a shift parameter (to which §§ 2, 3 are devoted) is very interesting, especially for d.f.'s $F(x)$ concentrated on a bounded interval.

Probably, in this last case the efficiency of sequential estimation as compared with estimation from a sample of fixed size is the same as in problems of hypothesis testing. Although in ⁽⁶⁾ it is indicated that the existence of the Wolfowitz inequality does not allow sequential estimates to be noticeably superior in quality to the best estimates constructed from a sample of fixed size, our results show that, at least for distributions $F(x - \theta)$, where $F(x)$ is concentrated on a finite interval, this is not so. The point is that in this case the conditions required in deriving the Rao-Cramér and Wolfowitz inequalities, generally speaking, are not satisfied, and other considerations are required.

1. Suppose that the d.f. $F(x)$ is concentrated on a bounded interval. It may always be assumed that this interval is $(-a, a)$.

In estimating the shift parameter θ , it is natural to use equivariant estimates $\bar{\theta}_n = \bar{\theta}_n(x_1, x_2, \dots, x_n)$, satisfying, for all $c \in R^1$, the condition

$$\bar{\theta}_n(x_1 + c, \dots, x_n + c) = c + \bar{\theta}_n(x_1, \dots, x_n).$$

Denote the class of equivariant estimates by $\mathfrak{R}$. For $\bar{\theta}_n \in \mathfrak{R}$, obviously, $E_\theta(\bar{\theta}_n - \theta)^2$ does not depend on θ , and therefore there exists an optimal estimate $\hat{\theta}_n$,

$$E_{\theta}(\hat{\theta}_n - \theta)^2 = \min_{\bar{\theta}_n \in \mathfrak{R}} E_{\theta}(\bar{\theta}_n - \theta)^2,$$

called the Pitman estimate.

Theorem 1. Let

$$F(x) = \int_{-a}^x f(u) du$$

and let the density $f(x)$, for some $c_1, c_2, \alpha_1, \alpha_2$, $c_1 > 0$, $c_2 > 0$, $-1 < \alpha_1 < 1$, $-1 < \alpha_2 < 1$, satisfy the conditions $f(x) \geq c_1(a+x)^{\alpha_1}$ in some neighborhood Δ_1 of the point $x = -a$, and $f(x) \geq c_2(a-x)^{\alpha_2}$ in some neighborhood Δ_2 of the point $x = a$. Then, if $\alpha = \max\{\alpha_1, \alpha_2\}$, then

$$E_{\theta}(\hat{\theta}_n - \theta)^2 \leq Cn^{-2/(1+\alpha)}, \quad (1)$$

where the constant C depends only on $a, c_1, c_2, \alpha_1, \alpha_2$ and Δ_1, Δ_2 .

This theorem gives, at least, a partial answer to the question posed by L. N. Bolshev about the behavior of the variance of the Pitman estimate depending on the form of the density $f(x)$ at the endpoints of the interval $(-a, a)$.

Note that, whatever the d.f. $F(x)$, the estimate

$$E_{\theta}(\hat{\theta}_n - \theta)^2 \leq c/n.$$

Our estimate (1) is, obviously, better than this trivial one.

2. We now turn to the scheme of sequential estimation of θ . Here the number of observations is determined by a stopping rule τ , and the estimates are statistics $\bar{\theta} = \bar{\theta}(x_1, \dots, x_{\tau})$.

It seems natural to consider—for estimating the location parameter—invariant stopping rules τ , i.e., such that the sets $\{\tau = n\}$, for $n \geq 2$, belong to the σ -algebra generated by the vector

$$Y_n = (x_2 - x_1, \dots, x_n - x_1);$$

we shall also assume that $\{\tau = 1\} = \emptyset$.

If, as before, we restrict ourselves to equivariant estimates, for which, for all $c \in R^1$,

$$\bar{\theta}(x_1 + c, \dots, x_{\tau} + c) = c + \bar{\theta}(x_1, \dots, x_{\tau}),$$

then for a stopping rule τ the optimal equivariant estimate $\hat{\theta}_\tau$ (it may be called the sequential Pitman estimate corresponding to the rule τ) has the following form: on the set $\{\tau = n\}$, $\hat{\theta}_\tau = \hat{\theta}_n$.

Put $x_{(n)} = \min\{x_1, \dots, x_n\}$ and $x^{(n)} = \max\{x_1, \dots, x_n\}$. Fix some small $\varepsilon > 0$ and consider the stopping rule $\tau = \tau_\varepsilon$:

$$\tau_\varepsilon = \min\{n : x^{(n)} - x_{(n)} > 2a - \varepsilon\}. \quad (2)$$

It is easy to see that τ_ε is an invariant rule and that, upon stopping at time τ_ε , we can determine the value of θ with accuracy up to ε .

Theorem 2. Under the conditions of Theorem 1, for the stopping rule (2) we have

$$E_\theta(\hat{\theta}_\tau - \theta)^2 \leq C(E_\theta\tau)^{-2/(1+\alpha)}, \quad (3)$$

where C depends on the same quantities as in Theorem 1.

Although estimate (3) is of the same order as (1) (the role of n in the sequential case is played by $E_\theta\tau$), it should be remembered that both are merely upper bounds for the variances of Pitman statistics. For particular distributions, $E_\theta(\hat{\theta}_\tau - \theta)^2$ turns out to be substantially smaller than $E_\theta(\hat{\theta}_n - \theta)^2$.

3. Suppose that $F(x)$ is the d.f. of the uniform law on $(-a, a)$, and denote $2a = \Delta$. In this case

$$\hat{\theta}_n = (x_{(n)} + x^{(n)})/2, \quad E_\theta(\hat{\theta}_n - \theta)^2 = \Delta^2/2(n+1)(n+2).$$

For rule (2) we have

$$E_\theta\tau = 2\Delta/\varepsilon, \quad E_\theta(\hat{\theta}_\tau - \theta)^2 = \varepsilon^2/24 = \Delta^2/6(E_\theta\tau)^2.$$

If we choose $\varepsilon = 2\Delta/n$, then the relative efficiency of the estimate $\hat{\theta}_\tau$ in comparison with $\hat{\theta}_n$ will be

$$\text{reff}(\hat{\theta}_\tau, \hat{\theta}_n) = E_\theta(\hat{\theta}_n - \theta)^2 / E_\theta(\hat{\theta}_\tau - \theta)^2 = 3n^2/(n+1)(n+2) \xrightarrow{n \rightarrow \infty} 3.$$

Apparently, sequential Pitman estimates for the location parameter of a d.f. $F(x - \theta)$ concentrated on a bounded interval are, in many cases, appreciably better than estimates constructed from a sample of fixed size.

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Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.

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