

# ON THE ASYMPTOTICS OF THE PREDICTION ERROR IN THE MULTIDIMENSIONAL CASE

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE ASYMPTOTICS OF THE PREDICTION ERROR IN THE MULTIDIMENSIONAL CASE

(Presented by Academician Yu. V. Linnik on 8 VII 1968)

1. Consider a stationary, in the broad sense, sequence  $x(n) = (x^1(n), \dots, x^m(n))$ . With this sequence there is naturally associated a sequence of subspaces  $X(n)$ , generated by the random variables  $x^1(n), \dots, x^m(n)$ . Denote by  $H_a^b$  the closed, in mean square, linear span of the subspaces  $X(s)$ ,  $a \leq s \leq b$ . The linear space  $H_{-\infty}^{\infty}$  is a Hilbert space with scalar product  $(x, y) = Ex\bar{y}$ .

For two subspaces  $H_{\varphi}, H_{\psi}$  of the space  $H_{-\infty}^{\infty}$ , put

$$\tau(H_{\varphi}, H_{\psi}) = \sum_{k,s} |(\varphi_k, \psi_s)|^2,$$

where  $\{\varphi_k\}, \{\psi_s\}$  are orthonormal bases in the subspaces  $H_{\varphi}, H_{\psi}$ , respectively.

Let  $\bar{H}(n)$  be the orthogonal complement in the space  $H_{-\infty}^{\infty}$  to the subspace  $H_n^{-1}$ . Put  $\tau_n = \tau(X(0), \bar{H}(n))$ . The quantity  $\tau_n$  can serve as a measure of the accuracy of the linear prediction of vectors  $\xi = (\xi^1, \dots, \xi^m)$  with coordinates  $\xi^i \in X(0)$  from the past of length  $n$ , i.e., from the random variables  $x^i(s)$ ,  $i = 1, \dots, m$ ;  $s = -1, \dots, -n$ . In particular, if the random variables  $\xi^1, \dots, \xi^m$  form a basis in  $X(0)$ , then there are constants, independent of  $n$ ,  $0 < m \leq M < \infty$  such that  $m\tau_n \leq \sigma_n^2(\xi) \leq M\tau_n$ , where  $\sigma_n^2(\xi) = \sum_{i=1}^m \sigma_n^2(\xi^i)$ , and  $\sigma_n^2(\xi^i)$  is the square of the error of the linear prediction of the random variable  $\xi^i$  from the past of length  $n$ .

If the sequence  $x(n)$  is linearly regular (see (1)), then the quantity  $\tau_{\infty} > 0$ . It is clear that  $\delta_n = \tau_n - \tau_{\infty} \geq 0$  and  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . We shall study the rate of decrease of the quantity  $\delta_n$  to zero depending on the properties of the spectral density (s.d.)  $f(\lambda)$  of a linearly regular sequence  $x(n) = (x^1(n), \dots, x^m(n))$  of full rank. In the one-dimensional case an analogous problem was considered by I. A. Ibragimov ((2), see also (3,4)).

**Theorem 1.** In order that, as  $n \rightarrow \infty$ ,

$$\delta_n = O(n^{-2p}), \quad p > 1/2, \quad p \text{ nonintegral}, \quad (1)$$

it is necessary and sufficient that the s.d.  $f(\lambda)$  almost everywhere coincide with a continuous matrix-function having a strictly positive determinant, all elements of this matrix having absolutely continuous derivatives  $f_{ij}^{(r-1)}(\lambda)$ ,  $r = [p]$ , the  $r$ -th derivatives  $f_{ij}^{(r)}(\lambda) \in L_2(-\pi, \pi)$  and satisfying there a Hölder condition of order  $\alpha = p - [p]$ ; the latter means that

$$\sup_{t \leq h} \left( \int_{-\pi}^{\pi} |f_{ij}^{(r)}(\lambda) - f_{ij}^{(r)}(\lambda + t)|^2 d\lambda \right)^{1/2} = O(h^\alpha). \quad (2)$$

2. The proof of Theorem 1 is based on the following lemma.

**Lemma 1.** *If condition (1) is satisfied, the collection  $x^i(s)$ ,  $i = 1, \dots, m$ ;  $s = 0, \pm 1, \dots$ , forms a Riesz basis in the space  $H_{-\infty}$ .*

We first derive Theorem 1 from Lemma 1. Let condition (1) be satisfied. Denote by  $\Psi^i(s)$ ,  $i = 1, \dots, m$ ;  $s = -1, \dots, -k, \dots$ , the system conjugate to the system  $x^i(s)$ ,  $i = 1, \dots, m$ ;  $s = -1, \dots, -k, \dots$ , in the space  $H_{-\infty}^{-1}$ . Let  $\widetilde{H}(n)$  be the orthogonal complement to the space  $H_{-n}^{-1}$  in the space  $H_{-\infty}^{-1}$ . Obviously,

$$\delta_n = \tau(X(0), \widetilde{H}(n)).$$

Since, by Lemma 1, the collection  $x^i(s)$ ,  $i = 1, \dots, m$ ;  $s = -1, \dots, -k, \dots$ , forms a Riesz basis in the space  $H_{-\infty}^{-1}$ , the collection  $\Psi^i(s)$ ,  $i = 1, \dots, m$ ;  $s = -1, \dots, -k, \dots$ , forms a Riesz basis in the space  $H_{-\infty}^{-1}$ . Consequently,

$$\delta_n \asymp \sum_{-s=n+1}^{\infty} \sum_{j=1}^m \sum_{i=1}^m |(x^i(0), \Psi^j(s))|^2, \quad (3)$$

where the symbol  $a_n \asymp b_n$  means that

$$0 < \liminf_n \frac{a_n}{b_n} \leq \limsup_n \frac{a_n}{b_n} < \infty.$$

It also follows from Lemma 1 (see (5)) that there exists an everywhere defined, bounded, invertible, positive Hermitian operator  $B$  in  $H_{-\infty}^{-1}$  such that

$$x^i(s) = B\Psi^i(s), \quad i = 1, \dots, m; \quad s = -1, -2, \dots \quad (4)$$

From (3) and (4) it follows that

$$\delta_n \asymp \sum_{-s=n+1}^{\infty} \sum_{j=1}^m \sum_{i=1}^m |(x^i(0), x^j(s))|^2.$$

Hence, using the results of approximation theory (see (6), Ch. V), one can show that the matrix  $f(\lambda)$  coincides almost everywhere with a continuous matrix whose elements have derivatives of the order indicated in Theorem 1, and that condition (2) is satisfied. In particular, the trace of the mentioned continuous matrix is a bounded function and, consequently, its determinant is strictly positive (see Lemma 1).

The proof of sufficiency is carried out analogously.

3. In the proof of Lemma 1 some results from the theory of orthogonal matrix polynomials are used (see also (7)).

Consider the space  $L_2^m(f)$  of matrices  $S(\lambda)$  of size  $m \times n$  such that

$$\|S\|_f^2 = \text{sp} \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\lambda) f(\lambda) S^*(\lambda) d\lambda < \infty.$$

Put

$$(S, P)_f = \frac{1}{2\pi} \int_{-\pi}^{\pi} S f P^* d\lambda.$$

The condition

$$(\varphi_n, \varphi_m)_f = \begin{cases} 0, & m \neq n, \\ E, & m = n, \end{cases} \quad (5)$$

uniquely determines a sequence of polynomials orthogonal in the sense of (5),  $\varphi_n(z) = k_n z^n + \dots$ , where  $k_n$  is a positive Hermitian matrix.

**Lemma 2.** *All zeros of the function  $\text{Det } \varphi_n(z)$  lie inside the unit circle.*

Put

$$s_n(x, z) = \sum_0^n \varphi_n^*(x) \varphi_n(z).$$

**Lemma 3.** Let  $g(z)$  be an arbitrary polynomial of degree  $n$ . Then

$$(g(z), s_n(x, z))_f = g(x), \quad z = e^{i\lambda}.$$

**Lemma 4.** The identity

$$s_n(0, z) = \sum_0^n \varphi_n^*(0) \varphi_n(z) = k_n z^n \varphi_n^*(z)$$

holds.

In particular,

$$s_n(0, 0) = \sum_0^n \varphi_n^*(0) \varphi_n(0) = k_n^2.$$

**Lemma 5.** The recurrence formula

$$k_n \varphi_{n+1}(z) = k_{n+1} z \varphi_n(z) + \varphi_{n+1}(0) z^n \varphi_n^*(z)$$

holds.

**Lemma 6.** Let  $g(z)$  be a polynomial of degree  $n$  with coefficient  $E$  at the leading term. Then

$$\text{Det}(g, g)_f \geq \text{Det } k_n^{-2}.$$

The polynomial  $k_n^{-1} \varphi_n(z)$  minimizes the quantity  $\text{Det}(g, g)_f$  among polynomials of degree  $n$  with coefficient  $E$  at the leading term.

The following Lemmas 7 and 8 are proved almost in the same way as the corresponding assertions in the one-dimensional case <sup>(2)</sup>.

**Lemma 7.** Under condition (1),

$$\sum |\varphi_k(0)|^2 < \infty.$$

**Lemma 8.** Under condition (1), there exist constants independent of  $n$ ,  $0 < m \leq M < \infty$ , such that

$$mE \leq s_n(0, z) s_n^*(0, z) \leq ME, \quad z = e^{i\lambda}.$$

Denote by  $G(z)$  the maximal analytic matrix inside the unit disk (see <sup>(1)</sup>) such that

$$f(\lambda) = G(e^{i\lambda}) G^*(e^{i\lambda}).$$

It is easy to show, using Lemmas 3 and 7, that under condition (1) the sequence  $s_n(0, z)$  converges on the unit circle to the matrix  $[G^{-1}(0)]^* G^{-1}(z)$  in the uniform metric. Hence, and from Lemma 8, follows the existence of constants  $0 < m \leq$

$M < \infty$  such that the relation  $mE \leq f(\lambda) \leq ME$  is satisfied for almost all  $\lambda \in [-\pi, \pi]$ . The obtained relation is equivalent to the assertion of Lemma 1.

4. Consider the case when  $\delta_n = O(e^{-cn})$ ,  $c > 0$ . The following two theorems are proved by the same method as Theorem 1 was proved (cf. Theorems 4 and 5 of <sup>(2)</sup>).

**Theorem 2.** The relation

$$\delta_n = O(e^{-cn}), \quad c > 0,$$

holds if and only if the s.p.  $f(\lambda)$  almost everywhere coincides with a matrix admitting analytic continuation into the strip of values of the complex argument  $z = \lambda + i\mu$  of width  $c$  and having a strictly positive determinant.

**Theorem 3.** The relation

$$\delta_n = O(e^{-cn})$$

holds for all  $c > 0$  if and only if the analytic continuation of the matrix mentioned in Theorem 2 is an entire function and its determinant is strictly positive.

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## REFERENCES

1. Yu. A. Rozanov, *Stationary Random Processes*, Moscow, 1963.
2. I. A. Ibragimov, *Theory of Probability and Its Applications*, 9, no. 4 (1964).
3. U. Grenander, M. Rosenblatt, *Trans. Am. Math. Soc.*, 76, 112 (1954).
4. G. Baxter, *Math. Scand.*, 10, 2 (1962).
5. N. K. Bari, *Uchen. zap. MGU*, 4, issue 148 (1951).
6. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960.
7. V. Gyires, *Magyar tud. acad. mat. kutato int. közl.*, 7, ser. A, 1–2 (1962).

*Note: Figure translations are in progress. See original paper for figures.*

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