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1969

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Abstract

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UDC 517.949.2+517.948.35

MATHEMATICS

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ON THE KERNELS OF SPECTRA OF FAMILIES OF OPERATORS

(Presented by Academician A. N. Tikhonov on 9 VII 1968)

Consider a family of operators $\{R_N\}$ mapping the normed spaces U_N of dimension N , $N = 1, 2, \dots$, into themselves. We shall assume that the operators R_N are uniformly bounded:

$$\|R_N\| < C. \quad (1)$$

Definition 1. In view of (1), we shall say that a complex number λ belongs to the resolvent set of the family of operators $\{R_N\}$ if there exist $\varepsilon > 0$ and N_0 such that for all N , $N > N_0$, and all u , $u \in U_N$, the inequality

$$\|R_N u - \lambda u\| \geq \varepsilon \|u\|$$

is satisfied.

By the spectrum of the family of operators $\{R_N\}$ we shall mean the complement of the resolvent set in the whole complex plane.

Theorem 1. *The spectrum of the family of operators $\{R_N\}$ is a closed set.*

Definition 2. Let $\{\varepsilon_N\}$ be some nonincreasing sequence of numbers tending to zero. We shall call the $\{\varepsilon_N\}$ -kernel of the spectrum of the family of operators $\{R_N\}$ the set $\Lambda_{\{\varepsilon_N\}}$, defined as follows:

$$\Lambda_{\{\varepsilon_N\}} = \bigcap_{k=1}^{\infty} \overline{\bigcup_{N>k} \Lambda_N},$$

where Λ_N is the set of points λ for which the inequality $\|R_N u - \lambda u\| \leq \varepsilon_N \|u\|$ has a nontrivial solution u , and $\overline{\bigcup \Lambda_N}$ is the closure of the set $\bigcup \Lambda_N$.

Theorem 2. *The $\{\varepsilon_N\}$ -kernel of the spectrum of the family of operators $\{R_N\}$ lies in the disk $|\lambda| \leq C$, where C is introduced in (1), is closed, nonempty, and belongs to the spectrum of the family of operators $\{R_N\}$.*

Theorem 3. *The totality of $\{\varepsilon_N\}$ -kernels of the spectrum, constructed for all possible nonincreasing sequences $\{\varepsilon_N\}$ converging to zero, coincides with the whole spectrum.*

Definition 3. Let $a, a \geq 1$, be a certain constant. We shall call the kernel with exponent a of the spectrum of the family of operators $\{R_N\}$ the intersection of the $\{\varepsilon_N\}$ -kernels, $\varepsilon_N = N^{-k}a^{-N}$, constructed for all natural k . The kernel with exponent $a = 1$ will sometimes be called the arithmetic kernel of the spectrum.

Definition 4. We shall call the absolute kernel of the spectrum of the family of operators $\{R_N\}$ the intersection of all $\{\varepsilon_N\}$ -kernels whatsoever.

Theorem 4. *The kernel with exponent $a, a \geq 1$, and the absolute kernel are closed. If $a_1 \geq a_2$, then the kernel with exponent a_1 is contained in the kernel with exponent a_2 . The absolute kernel coincides with the $\{\varepsilon_N\}$ -kernel for $\varepsilon_N \equiv 0$.*

Definition 5. We shall call the radius of the spectrum (the radius of the kernel) of the family of operators $\{R_N\}$ the radius of the smallest closed disk with center at the point $\lambda = 0$ that contains the spectrum (the kernel of the spectrum).

Theorem 5. For any $\varepsilon > 0$ there exists a number $A = A(\varepsilon)$, independent of N , such that

$$\|R_N^m\| \leq A(\varepsilon)(\rho + \varepsilon)^m, \quad m = 1, 2, \dots,$$

where ρ is the radius of the spectrum of the family of operators $\{R_N\}$.

Theorem 6. Let ρ be the radius of the $\{\varepsilon_N\}$ -kernel of the spectrum. Then for every $\varepsilon > 0$ there exists an index N_0 , $N_0 = N_0(\varepsilon)$, such that for $N > N_0$ the inequality

$$\|R_N^m\| \leq (\rho + \varepsilon)^{m+1}/\varepsilon_N, \quad m = 1, 2, \dots$$

holds.

Theorem 7. Let ρ be the radius of the $\{\varepsilon_N\}$ -kernel of the spectrum. Then, for any $\varepsilon > 0$ and any N_0 , there is an N , $N > N_0$, such that

$$\|R_N^m\| \geq (\rho - \varepsilon)^m \left(1 - \frac{\varepsilon_N(C + \varepsilon)^m}{(\rho - \varepsilon)^m} \left(\min m, \frac{1}{1 - (\rho + \varepsilon)/C} \right) \right),$$

where C is the number introduced in (1).

Corollary. From Theorems 3 and 7 there follows the known theorem ⁽¹⁾, asserting that, for the norms of the operators R_N^m to be bounded uniformly in m and N , it is necessary that the spectrum of the family of operators $\{R_N\}$ lie in the unit disk $|\lambda| \leq 1$.

Let us note that in Theorem 7 one cannot put $\varepsilon = 0$.

We shall discuss the question of to what extent the spectrum of a family of operators $\{R_N\}$ is independent of the choice of norms in the spaces U_N .

Theorem 8. The absolute kernel of the spectrum of a family of operators does not depend on the choice of norms in the spaces U_N .

Theorem 9. For a given family of operators $\{R_N\}$, whose eigenvalues are bounded in the aggregate, the sequence of norms in the spaces U_N can be chosen so that the operators R_N are bounded in the aggregate and so that the spectrum of the family of operators $\{R_N\}$ coincides with its absolute kernel.

Theorem 10. Let R_N be unitary operators in unitary spaces U_N . Then the spectrum of the family of operators $\{R_N\}$ lies on the unit circle and coincides with its absolute kernel.

Theorem 11. Let R_N be Hermitian operators in unitary spaces U_N , bounded in the aggregate. Then the spectrum of the family of operators $\{R_N\}$ is real and coincides with its absolute kernel.

Generally speaking, the spectrum of a family of operators $\{R_N\}$ bounded in the aggregate is not exhausted by its absolute kernel, as we shall show below, relying on the example from ⁽¹⁾, Ch. VI, § 1. Comparing this with Theorem 9, we see that the spectrum of a family of operators depends, in general, on the choice of norms in the spaces U_N .

Theorem 12. Suppose that for each N two norms $\|\cdot\|_N^{(1)}$ and $\|\cdot\|_N^{(2)}$ are given in the space U_N , and suppose that there exists a number s , independent of N , such that for all sufficiently large values of N the inequalities

$$\begin{aligned} \sup_{\|u\|^{(1)}=1} \|u\|_N^{(2)} &\leq N^s \inf_{\|u\|^{(1)}=1} \|u\|_N^{(2)}, \\ \sup_{\|u\|^{(2)}=1} \|u\|_N^{(1)} &\leq N^s \inf_{\|u\|^{(2)}=1} \|u\|_N^{(1)}. \end{aligned} \quad (2)$$

hold.

Suppose, further, that the operators R_N are bounded in the aggregate in each of these norms. Then the kernels, with any exponent a , $a \geq 1$, of the spectrum of the family of operators $\{R_N\}$ in both sequences of norms coincide.

Condition (2) is satisfied for any pair of norms ordinarily used in the theory of difference schemes:

$$\|u\| = \max |u_k|; \quad \|u\| = \left(\sum_k |u_k|^p \right)^{1/p}$$

and so on.

Example. Consider the spectrum of the family of operators $\{R_N\}$ ((1), Ch. VI, § 1). Define the operator R_N as follows: this operator assigns to an $(N+1)$ -dimensional vector $u = (u_0, u_1, \dots, u_N)$, with norm $\|u\| = \max |u_k|$, a certain vector $v = (v_0, v_1, \dots, v_N)$ according to the formulas

$$v_n = (1 - \xi)u_n + \xi u_{n+1}, \quad n = 0, 1, \dots, N-1; \quad v_N = 0,$$

where ξ is a certain positive constant. This family of operators arises when replacing the boundary-value problem for the equation $u_t - u_x = 0$ by a difference analogue. In (1) it is shown that the spectrum of the family of operators $\{R_N\}$ consists of the closed disk of radius ξ with center at the point $\lambda = 1 - \xi$ and of the point $\lambda = 0$. It can be shown that the kernel with exponent a consists of the closed disk with center at the same point $\lambda = 1 - \xi$ and radius ξ/a , and also of the point $\lambda = 0$. Taking $a = 1$, we see that the arithmetic kernel of the spectrum coincides with the entire spectrum. The absolute kernel of the spectrum of this family of operators consists of the two points $\lambda = 0$ and $\lambda = 1 - \xi$.

One can construct an example of such a family of operators for which the arithmetic kernel of the spectrum does not coincide with the entire spectrum.

Received
4 VI 1968

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Note: Figure translations are in progress. See original paper for figures.

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