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Abstract

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MATHEMATICS

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BASES OF EIGENVECTORS OF COMPLETELY NONUNITARY CONTRACTIONS

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In the present note we study systems of eigenvectors of completely nonunitary contractions (see also ⁽¹⁾).

0.1. Let E be a Hilbert space, R the ring of bounded operators in E ; $H^2(E)$ the space of all functions, regular in the unit disk $D = \{z : |z| < 1\}$, with values in E and with norm

$$\|f\|_2^2 \stackrel{\text{def}}{=} \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{it})\|_E^2 dt < \infty.$$

Further, let S be the shift operator in $H^2(E)$

$$(Sf)(z) = zf(z), \quad |z| < 1, \quad f \in H^2(E);$$

S^* the adjoint operator, and T the restriction of S^* to some S^* -invariant subspace of the form $K = H^2(E) \ominus \Theta H^2(E)$. Here Θ is a regular bounded function in D with values in R , unitary almost everywhere on the unit circle (an inner ^(2,3) operator-function).

It was shown in ⁽²⁾ that every linear operator A acting in a Hilbert space and satisfying the conditions $\|A\| \leq 1$, $\text{s-lim}_{n \rightarrow \infty} A^n = \text{s-lim}_{n \rightarrow \infty} A^{*n} = 0$, is unitarily equivalent to some operator of the form T . The corresponding function Θ , up to inessential factors, coincides with the characteristic function of the operator A^* (see ^(2,4)).

0.2. We shall investigate the system of eigenfunctions (e.f.) of the operator T under the same assumptions as in the note ⁽¹⁾: the spectrum of the operator T in the disk D consists of isolated points $\bar{z}_k$, $k = 1, 2, \dots$, which are simple poles of the resolvent, and the system of e.f. of T is complete in K .

As in ⁽¹⁾, let π_k (respectively Δ_k) be the projector onto the kernel of the operator $\Theta(z_k)$ (respectively $\Theta^*(z_k)$), $|z_k| < 1$. Under the assumptions made (see ^(5,6)) Θ factors in the form $\Theta = \Theta_k B_k$, where both factors are inner and

$$B_k(z) = \frac{z_k - z}{1 - \bar{z}_k z} \frac{\bar{z}_k}{|z_k|} \pi_k + (1 - \pi_k), \quad |z| < 1,$$

$\bar{z}_k$ is an eigenvalue of T , and $\Theta_k(z_k)$ is an isomorphism in E . Hence it follows, in particular, that $\dim \text{Ker } \Theta(z_k) = \dim \text{Ker } \Theta^*(z_k)$.

Let e_k^i be any orthonormal basis in $\pi_k E$. The system of (all) e.f. of the operator T (respectively T^*) corresponding to the eigenvalue $\bar{z}_k$ (respectively z_k) is formed by the functions

$$\begin{aligned} \varphi_k^i &= (1 - \bar{z}_k z)^{-1} \Theta_k^{*-1}(z_k) e_k^i, \\ \psi_k^i &= (1 - \bar{z}_k z)^{-1} \Theta_k(z) e_k^i. \end{aligned} \quad (1)$$

The correspondingly normalized systems $\{\varphi_k^i\}, \{\psi_k^i\}$ are biorthogonal in $H^2(E)$ (see (1)). In what follows we shall assume that the system

$\{\varphi_k^i\}$ is uniformly minimal, i.e.

$$\sup_{k,i} \|\Theta_k^{*-1}(z_k) e_k^i\| < \infty. \quad (2)$$

Let us note that condition (2) entails the ω -linear independence⁽⁶⁾ of the system $\{\varphi_k^i\}$.

1.1. Theorem. Under assumptions 0.2 the following conditions are equivalent:

1. The system (1) of root vectors of the operator T forms a Bari basis.
2. The infinite product converges (cf. (8))

$$\prod_{k,i} \|\Theta_k^{*-1}(z_k) e_k^i\|.$$

3.

$$\sum_k \|D_{\Theta_k^*} \Delta_k\|_{\gamma_2}^2 < \infty,$$

where

$$D_{\Theta_k^*} = [I - \Theta_k(z_k) \Theta_k^*(z_k)]^{1/2}.$$

4. The Gram matrix

$$\{\delta_{ki,lj} - (\varphi_k^i, \varphi_l^j) \|\varphi_k^i\|^{-1} \|\varphi_l^j\|^{-1}\}$$

generates in l^2 a Hilbert-Schmidt operator.

1.2. Theorem. Suppose that assumptions 0.2 are satisfied and that the system of eigenspaces (e.s.) of the operator T is uniformly minimal, i.e. (1)

$$\sup_k \|\Delta_k \Theta_k^{*-1}(z_k) \pi_k\| < \infty.$$

The system of e.s. forms a Bari basis ⁽⁶⁾, if the condition

$$\sum_{\substack{i,k \\ i \neq k}} (1 - |z_i|^2)(1 - |z_k|^2) |1 - \bar{z}_i z_k|^{-2} \|\Delta_i \Delta_k\|_R^2 < \infty$$

is satisfied.

From Theorems 1.1 and 1.2 one easily obtains the known spectral characterizations of Bari bases of root vectors of contraction operators ^(7,8).

2.1. Let the eigenvalues of the operator T have only one limit point $z = 1$, and suppose that for some a , $a < \pi$, the condition

$$|\arg((I - T)f, f)| < a, \quad f \in K. \quad (3)$$

is fulfilled. Further, let

$$\delta_k = \inf_{i, i \neq k} \left\{ |z_k - z_i| |1 - \bar{z}_k z_i|^{-1} \right\}^{m_i}, \quad m_k = \dim \text{Ker } \Theta(z_k).$$

2.2. Theorem. Suppose that assumptions 0.2 and (3) are satisfied. If

$$0 < \prod_k \delta_k^{m_k} < \infty, \quad (4)$$

then the system of root vectors of the operator T is a Bari basis. In the case $\dim E = 1$, condition (4) is also necessary.

The proof of this assertion is obtained from Theorem 1.1 and certain estimates ⁽⁹⁾ for Blaschke products with zeros lying in an angle.

3.1. We now consider an operator T whose sequence of eigenvalues decomposes into a certain number of Carleson subsequences. Recall ^(6,8,10) that a Carleson sequence $\{\zeta_k\}_{k=1}^\infty$, $|\zeta_k| < 1$, is characterized by the property

$$\delta = \inf_k \prod_{i \neq k} \left| \frac{\zeta_i - \zeta_k}{1 - \bar{\zeta}_i \zeta_k} \right| > 0. \quad (C)$$

In works ^(1,8) it was shown that the system of root vectors of the operator T forms a Riesz basis ⁽⁷⁾, if the sequence of its eigenvalues $\{\bar{z}_k\}$ is Carleson.

Assume that the sequence $\{\bar{z}_k\}_{k=1}^\infty$ of eigenvalues of the operator T is the union of a finite number of Carleson sequences $\{z_n^{(i)}\}_{n=1}^\infty$, $i = 1, 2, \dots, m$. Then (see ^(1,6,10)) the matrices

$$\left\{ \frac{(1 - |z_l^{(i)}|^2)^{1/2} (1 - |z_n^{(i)}|^2)^{1/2}}{1 - \bar{z}_l^{(i)} z_n^{(i)}} \right\}_{l,n=1}^\infty, \quad i = 1, 2, \dots, m,$$

generate positive definite operators in l^2 , whose lower and upper bounds (γ_i, Γ_i) , respectively) depend only on the constant $\delta^{(i)}$ in condition (C).

Suppose, moreover, that there exist orthoprojectors $P^1, P^2, \dots, P^m$ in E such that

$$P_{\text{Ker } \Theta^*(z_k^{(i)})} \equiv \Delta_k^{(i)} \rightarrow P^i$$

in norm as $k \rightarrow \infty$, $i = 1, 2, \dots, m$.

Then we shall say that the system of eigenfunctions of the operator T decomposes into m Carleson series $\{K(P^i, \gamma^i, \Gamma^i)\}$, $i = 1, 2, \dots, m$.

3.2. Theorem. *The system of eigenfunctions of the operator T forms a Riesz basis if conditions 0.2, 3.1 are satisfied and*

$$\min_i \gamma_i > \max_i \sum_{j, j \neq i} a^{ij} \Gamma_i^{1/2} \Gamma_j^{1/2}. \quad (5)$$

Here $a^{ij} = \cos(P^{iE} \wedge P^{jE})$, $i, j = 1, 2, \dots, m$.

3.3. Remark. If $a^{ij} = 0$, $i \neq j$, then condition (5) is also necessary.

The following assertion, in a certain sense, is the converse of Theorem 3.2.

3.4. Theorem. *Suppose that conditions 0.2 are satisfied, the system of eigenfunctions of the operator T forms a Riesz basis, $\dim \Delta_{kE} = 1$, $k = 1, 2, \dots$, and the projectors Δ_k , $k \geq 1$, form a compact family. Then the set of eigenvalues $\{\bar{z}_k\}$ is the union of a finite number of Carleson sequences.*

3.5. Corollary. *If $\dim E < \infty$, and the system of eigenfunctions of the operator T forms a Riesz basis, then the sequence $\{\bar{z}_k\}$ is the union of a finite number of Carleson sequences.*

4.1. Applying Theorems 1.1 and 1.2 and some results of Remark ⁽¹⁾ in the case $\dim E = 1$, we arrive at criteria for the basis property of certain known systems of functions. Consider, for example, the system of powers $\varphi_k(x) = x^{\lambda_k}$, $x \in (0, 1)$, $\operatorname{Re} \lambda_k > -1/2$, $k = 1, 2, \dots$, in the space $L_2(0, 1)$. In order that the system φ_k form a Riesz basis in the closure of its linear span, it is necessary and sufficient that, for the numbers $\mu_k = i(\lambda_k + 1/2)$, the Carleson condition for the half-plane

$$\inf_k \prod_{l, l \neq k} \left| \frac{\mu_k - \mu_l}{\mu_k - \bar{\mu}_l} \right| > 0. \quad (6)$$

be satisfied. If $\operatorname{Im} \lambda_k = 0$, then (6) becomes the lacunarity condition obtained in ⁽¹¹⁾. If the λ_k lie in the angle $|\arg(\lambda_k + 1/2)| \leq \alpha < \pi$, then (6) is equivalent (see ⁽⁹⁾) to the condition

$$\inf_{k, l, l \neq k} \left| \frac{\mu_k - \mu_l}{\mu_k - \bar{\mu}_l} \right| > 0.$$

Necessary and sufficient conditions for the quadratic closeness of the basis $\{\varphi_k\}$ to an orthonormal one follow at once from Theorems 1.1 (or 1.2) and 2.2. Everything said applies equally to the system $\psi_k(x) = \exp i\mu_k x$, $x \in (0, \infty)$, $k = 1, 2, \dots$, in the space $L_2(0, \infty)$, and to the system $\chi_k(z) = (z - \mu_k)^{-1}$, $\operatorname{Im} z > 0$, $k = 1, 2, \dots$, in the Hardy space H^2 in the half-plane ⁽¹⁰⁾.

* In other words, the defect operators D_T, D_{T^*} are finite-dimensional.

In conclusion, we note that analogues of the results of this note can also be obtained for dissipative operators, if the disk D is replaced by the upper half-plane.

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