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Abstract

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MATHEMATICS

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ON THE QUESTION OF THE DEPENDENCE OF HOLOMORPHIC DIFFERENTIALS ON THE MODULI OF RIEMANN SURFACES

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1. Along with the study by many authors of the nature of the parameters determining the conformal structure of a Riemann surface—the moduli—the question of the character of the dependence on these moduli of various quantities connected with the surface has also been intensively investigated: differentials, Weierstrass points, etc. Various relations have been obtained here, mainly in terms of the theory of theta functions, which are very complicated. The introduction of a complex-analytic structure in the corresponding space of Riemann surfaces gives another approach to the solution of this question (see, for example, ^(2,3,6)).

In this note we formulate a theorem that gives a concrete basis in the vector space of holomorphic quadratic differentials (automorphic forms of weight (-4)) on every closed surface of genus $g > 1$, depending holomorphically on the moduli, and thereby gives, in a certain sense, an explicit form of the holomorphic continuation of such differentials into moduli space.

2. Let U_1 be the disk $|z| < 1$; U_2 the domain $1 < |z| \leq \infty$; Γ a fixed Fuchsian group of the first kind such that $S = U_2/\Gamma$ is a closed Riemann surface of genus $g > 1$, with Γ chosen in accordance with what is stated in ⁽⁴⁾, § 2. On the surface S we fix a class of homeomorphisms of S onto itself, for example, one containing the identity mapping.

Let $B_1(\Gamma)$ be the set of functions $\mu(z)$, measurable in the z -plane, such that $\mu(z) d\bar{z}/dz$ is invariant with respect to Γ , $\mu(z) = 0$ for $z \in U_1$, and $\|\mu\|_{L_\infty(U_2)} < 1$. For each $\mu \in B_1(\Gamma)$ there exists a unique quasiconformal homeomorphism $w = f_\mu(z)$ of the extended complex z -plane onto itself satisfying the Beltrami equation $w_{\bar{z}} = \mu w_z$ and normalized by the conditions $w(0) = 0$, $w'(0) = 1$, $w(1) = 1$. Then $\Gamma^\mu = f_\mu \Gamma f_\mu^{-1}$ is a quasifuchsian group of fractional-linear mappings of the w -plane, and $S^\mu = f_\mu(U_2)/\Gamma^\mu = f_\mu(S)$ is a closed Riemann surface of genus $g > 1$. Letting μ run through the entire set $B_1(\Gamma)$, we obtain all points $[S^\mu]$ of the Teichmüller space $T(S) = T_g$ (see, for example, ^(1,3,5,7)). We shall call functions μ and ν equivalent if they determine one and the same point of the

space T_g . Using the results of ^(1,3), we obtain that functions μ and ν from $B_1(\Gamma)$ are equivalent if and only if $f_\mu(z) = f_\nu(z)$ for $z \in \overline{U_1}$.

As moduli of the points $[S^\mu]$ of the space $T(S)$ one can take the corresponding $3g$ coefficients among the coefficients $a = (a_2, a_3, \dots, a_{3g-1})$ of the expansions

$$f_\mu(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U_1,$$

which fill a bounded domain D_{3g-3} on an analytic surface in $\mathbb{C}^{3g-2}$. These coefficients depend only on the equivalence classes $[\mu]$. Moreover, in each equivalence class $[\mu]$ there exists a function $\mu_a(z) \in B_1(\Gamma)$, depending holomorphically on these a as an element of $L_\infty(U_2)$, and the mapping $\tau : \mu_a \rightarrow a$ is biho-

locally homeomorphic homeomorphism from $L_\infty(U_2)$ to C^{3g-2} , generating a well-defined biholomorphic homeomorphism of the space $T(S)$ onto the domain D_{3g-3} . We shall denote the corresponding $3g$ coefficients—moduli—by the symbol $\hat{a} = (a_2, \dots, a_{3g-1})$, and the corresponding indices by the symbol $1, \dots, 3g-2$.

3. We identify the space $T(S)$ with the domain D_{3g-3} . The domain $f_\mu(U_2)$, determined by a point $a = (a_2, a_3, \dots, a_{3g-1}) \in D_{3g-3}$, and the quasifuchsian group $\Gamma^\mu = f_\mu \Gamma f_\mu^{-1}$ will be denoted by $D_2(a)$ and Γ_a , respectively. By $w_a = f_a(z)$ we shall denote the mapping $f_\mu(z)$ for $\mu(z) = \mu_a(z) = \tau^{-1}(a) \in [\mu]$.

For the mappings $w_{\mu+\nu} = f_{\mu+\nu}(z)$, for $\|\nu\|_{L_\infty(U_2)} \leq \varepsilon < \varepsilon_0$, $0 < \varepsilon_0 < 1$, the variational formula holds

$$w_{\mu+\nu} = w_\mu - \frac{w_\mu^2(w_\mu - 1)}{\pi} \iint_{w_\mu(U_2)} \frac{\theta(\zeta) d\sigma(\zeta)}{\zeta^2(\zeta - 1)(\zeta - w_\mu)} + O(\varepsilon^2), \quad (1)$$

where the estimate of the remainder term is uniform for $|w_\mu| \leq R < r_0(\varepsilon_0)$, $r_0(\varepsilon_0)$ is a completely determined function of ε_0 such that $\lim_{\varepsilon_0 \rightarrow 0} r_0(\varepsilon_0) = \infty$, and

$$\theta(w_\mu) = \left(\frac{\nu}{1 - \mu(\mu + \nu)} \frac{(w_\mu)_z}{(w_\mu)_z} \right) \circ (w_\mu)^{-1}. \quad (2)$$

It follows from this, in particular, that the mappings $f_a \circ f_{a_0}^{-1}(w)$ depend holomorphically on the moduli $a \in T(S)$ for fixed a_0 and w . Therefore the transformations $A_a = f_a \circ A \circ f_a^{-1}$, $A \in \Gamma$, of the group Γ_a (i.e. their coefficients) also depend holomorphically on the moduli a .

We shall also denote by $B(D_2(a), \Gamma_a)$ the complex Banach space of functions $\psi(w)$ holomorphic in the domain $D_2(a)$ such that $\psi(Aw)A'^2(w) = \psi(w)$, $A \in \Gamma_a$, and $\psi(w) = O(|w|^{-4})$ near $w = \infty$, with norm $\|\psi\| = \sup_{w \in D_2(a)} \lambda_a^{-2}(w) |\psi(w)|$, where $\lambda_a(w)|dw|$ is the Poincaré metric of the domain $D_2(a)$.

We construct over $T(S)$ the fibered space

$$\tilde{T}(S) = \{(a, w) \mid a \in T(S), w = f_a(z) \in D_2(a)\}.$$

The space $\tilde{T}(S)$ has a natural complex structure. Its base is $T(S) = D_{3g-3}$, and the projection $(a, w) \mapsto a$ is defined so that the fiber over the point $a \in T(S)$ is the domain $D_2(a)$ —the universal covering surface of

$$S_a = D_2(a)/\Gamma_a.$$

4. Put now

$$\psi_j(w, a) = \frac{1}{(j+1)!} \sum_{A \in \Gamma_a} \frac{d^{j+1}}{dz^{j+1}} \left[\frac{f_a^2(z)(f_a(z)-1)}{Aw - f_a(z)} \right] \Big|_{z=0} \frac{A'^2(w)}{(Aw)^2(Aw-1)}, \quad (3)$$

$$j = 1, 2, \dots, 3g-3.$$

Theorem. The functions $\psi_{i_j}^j(w, a)$, $j = 1, 2, \dots, 3g-3$, $i_j = 1, \dots, \widehat{3g-2}$, for $w = f_a(z)$ are holomorphic in w and a in $\tilde{T}(S)$, and for each fixed $a \in T(S)$ form a basis in the space $B(D_2(a), \Gamma_a)$.

The holomorphy of the functions (3) in w and a is established by estimating the terms of these series. To prove their linear independence, one uses the variational formula (1), (2), as well as the connection between the moduli

$$\hat{a} = (a_2, a_3, \dots, \widehat{a_{3g-2}}) \in T(S)$$

and the global moduli of L. Ahlfors—L. Bers (see $(^1, ^3, ^4)$).

This theorem gives a simultaneous uniformization of all holomorphic quadratic differentials on the surfaces S_a . The formula

$$\psi(w, a) = \sum_{j=1}^{3g-3} \xi_j \psi_{i_j}(w, a), \quad \xi_j = \text{const}, \quad i_j = 1, 2, \dots, \widehat{3g-2},$$

gives a “holomorphic pro-

“continuation” in $\tilde{T}(S)$ of each holomorphic quadratic differential on any surface S_a , $a_0 \in T(S)$.

The ratios $\psi_j(w, a)/\psi_i(w, a)$, $i \neq j$, are meromorphic functions on $\tilde{T}(S)$ and are rational for each $a \in T(S)$.

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