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Abstract

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MATHEMATICS

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ELLIPTIC BOUNDARY-VALUE PROBLEMS FOR PLANE DOMAINS WITH CORNERS AND DISCONTINUITIES

REACHING THE BOUNDARY

(Presented by Academician I. N. Vekua, 20 I 1969)

In the present note we state existence theorems for boundary-value problems satisfying the Lopatinskii condition in weighted spaces, analogous to those considered in ⁽¹⁾.

1°. Notation, formulation of the problem. Let Ω be a bounded plane domain with piecewise smooth boundary Γ , containing a finite number of nonzero angles, divided by a finite number of smooth curves γ_ν into parts Ω_μ . In Ω_μ let operators $L_\mu(x, y, \partial/\partial x, \partial/\partial y)$ be given, and on Γ, γ_ν , respectively, operators $B(x, y, \partial/\partial x, \partial/\partial y)$ and $C_\nu^\pm(x, y, \partial/\partial x, \partial/\partial y)$ in the form of matrices of dimensions respectively $N \times N$, $m \times N$, and $2m \times N$ (the number m will be defined below), whose elements $l_{ij}^{(\mu)}$, b_{ij} , and $c_{ij\nu}^\pm$ are homogeneous linear differential operators of orders respectively $s_i + t_j$, $\sigma_i + t_j$, $\sigma_i + t_j$, $\max_i s_i = 0$ (see ⁽²⁾). The operators L_μ are uniformly elliptic and satisfy the additional condition on L (see ⁽²⁾). Then the number m introduced above is equal to half the order of $\det L_\mu$. The coefficients entering into $l_{ij}^{(\mu)}$, b_{ij} , $c_{ij\nu}^\pm$ are assumed sufficiently smooth (at angular points and points of intersection of the curves this assumption can be weakened).

Consider the problem (see ⁽³⁾)

$$L_\mu(x, y, \partial/\partial x, \partial/\partial y)U = f(x, y), \quad (x, y) \in \Omega_\mu; \quad (1)$$

$$B(x, y, \partial/\partial x, \partial/\partial y)U = \varphi(x, y), \quad (x, y) \in \Gamma; \quad (2)$$

$$C_\nu^+ U^+ - C_\nu^- U^- = \psi_\nu(x, y), \quad (x, y) \in \gamma_\nu, \quad (3)$$

in the functional spaces

$$\|U_j\|_{\dot{W}_a^k(\Omega)}^2 = \sum_{|\alpha| \leq k} \iint_{\Omega} r^{a-2(k-|\alpha|)} |D^\alpha U_j| dx dy, \quad j = 1, \dots, N,$$

$$\|\varphi\|_{\dot{W}_a^{k-1/2}(\partial\omega)} = \inf_{\Pi} \|\Pi\varphi\|_{\dot{W}_a^k(\omega)},$$

where $\Pi\varphi \in \dot{W}_a^k(\omega)$ is an extension of φ from the boundary $\partial\omega$ into the domain ω , and r is the product of the distances from the point (x, y) to all angular points and points of intersection of the pairs (Γ, γ_ν) and (γ_μ, γ_ν) . The direct product of the spaces

$$\dot{W}_a^{k-s_1} \times \dots \times \dot{W}_a^{k-s_N} \times \dot{W}_a^{k-\sigma_1-1/2} \times \dots \times \dot{W}_a^{k-\sigma_m-1/2}$$

will be denoted by $\dot{H}_a^k$. Suppose that the operators $B, C_\nu^\pm$ jointly cover the operators L_μ in the sense of ⁽³⁾.

2°. Problems with one angular point. First consider problem (1)–(2) in a domain not divided by the curves γ_ν , with boundary Γ containing only one angular point (for example, the origin), and pose the auxiliary problem

$$L(0, 0; \partial/\partial x, \partial/\partial y)U = f, \quad (x, y) \in \Omega_0; \quad (4)$$

$$B(0, 0, \partial/\partial x, \partial/\partial y)U = \varphi, \quad (x, y) \in \Gamma_0, \quad (5)$$

where Γ_0 consists of two rays issuing from the angular point tangent to Γ , and Ω_0 –

the interior of the angle between them. After the change of variables

$$x = e^{-\tau} \cos \omega, \quad y = e^{-\tau} \sin \omega, \quad U_j = e^{-t_j \tau} V_j(\omega, \tau), \quad j = 1, \dots, N,$$

and the Fourier transform with respect to τ

$$\tilde{F}(\lambda) = \int_{-\infty}^{+\infty} F(\tau) e^{-i\lambda\tau} d\tau$$

the auxiliary problem (4)–(5) is transformed into a boundary-value problem in the strip $0 < \omega < \omega_0$ for a system of ordinary differential equations with respect to the functions $V_j(\omega, \tau)$, $j = 1, \dots, N$,

$$\tilde{L}(\omega, d/d\omega, i\lambda)\tilde{V} = \tilde{F}(\omega, \lambda); \quad (6)$$

$$\tilde{B}(\omega, d/d\omega, i\lambda)\tilde{V}|_{\omega=0; \omega_0} = \tilde{\Phi}(\lambda), \quad (7)$$

where $\tilde{L}$ and $\tilde{B}$ are the Fourier transforms with respect to τ of matrices consisting, respectively, of the elements

$$e^{-(s_i+t_j)\tau}l_{ij}(0, 0, \partial/\partial x, \partial/\partial y), \quad e^{-(\sigma_i+t_j)\tau}b_{ij}(0, 0, \partial/\partial x, \partial/\partial y)$$

and, by construction of the system, independent of τ . If problem (4)–(5) is properly elliptic and satisfies the Lopatinskii condition, then problem (6)–(7) satisfies conditions analogous to Conditions I and II for systems formulated in (4) for the case $s_i = 0$, $t_j = 2m$. Applying the method set forth in (4), one can show that for sufficiently large λ there exists a resolving operator $R(\lambda)$, acting from the intersection of the Sobolev spaces

$$W_2^{k-s_1} \times \dots \times W_2^{k-s_N} \times W_2^{k-\sigma_1-1/2} \times \dots \times W_2^{k-\sigma_m-1/2}$$

to the space

$$H^k = W_2^{k-t_1} \times \dots \times W_2^{k-t_N},$$

which is a meromorphic function of λ and is such that $\tilde{V} = R(\lambda)(\tilde{F}, \tilde{\Phi})$ is the solution of problem (6)–(7) satisfying the estimate

$$\begin{aligned} \int_0^{\omega_0} \sum_{q=0}^{k+t_j} [1 + |\lambda|^{2(k+t_j-q)}] \left| \frac{d^q \tilde{V}_j}{d\omega^q} \right|^2 d\omega \leq C \left\{ \int_0^{\omega_0} \sum_{i=1}^N \left(\left| \frac{d^{k-s_i} \tilde{F}_i}{d\omega^{k-s_i}} \right|^2 + |\lambda|^{k-s_i} |\tilde{F}_i|^2 \right) d\omega + \right. \\ \left. + \sum_{i=1}^m |\lambda|^{2(k-\sigma_i)-1} (|\tilde{\Phi}_i|_{\omega=0}|^2 + |\tilde{\Phi}_i|_{i=\omega_0}|^2) \right\}. \end{aligned} \quad (8)$$

Theorem 1. Suppose that the resolving operator $R(\lambda)$ for problem (4)–(5) has no poles on the line $\text{Im } \lambda = h$. Then for $(f, \varphi) \in \dot{H}_{2(k-h-1)}^k$ there exists a unique solution of problem (4)–(5), satisfying the estimate

$$\|U_j\|_{\dot{W}_{2(k-h-1)}^{k+t_j}} \leq C \|f, \varphi\|_{\dot{H}_{2(k-h-1)}^k}. \quad (9)$$

Theorem 2. If $R(\lambda)$ for the auxiliary problem (4)–(5) has no poles on the line $\text{Im } \lambda = h$, then for $(f, \varphi) \in \dot{H}_{2(k-h-1)}^k$, under the fulfillment of a finite number

of conditions of the type of functionals in $\dot{H}_{2(k-h-1)}^k$, $F_p(f, \varphi) = 0$, there exists a solution of problem (1)–(2)

$$U_j \in \dot{W}_2^{k+t_j}_{2(k-h-1)}, \quad j = 1, \dots, N.$$

If this solution is unique, then the estimate (9) is valid for it.

3°. Adjoint problems. We now consider problem (1)–(3) with one line of discontinuity γ , intersecting Γ at nonzero angles at the origin and at the point Q . Rotate the coordinate axes so that the x -axis becomes tangent to Γ , and the y -axis coincides with the inward normal. Then the equations of the curves Γ, γ in a neighborhood of the origin can be represented in the form $y = y_\Gamma(x)$, $y = y_\gamma(x)$. Since the angle between Γ and γ is nonzero, the transformation $x_1 = y - y_\Gamma$, $y_1 = y - y_\gamma$ is a smooth nondegenerate one, carrying problem (1)–(3) into the corresponding problem for the half-plane. Putting further, for $y < 0$, $z = -y$, $U_j(x, -z) = V_j(x, z)$, we see that,

that problem (1)–(3) is equivalent to the “doubled” boundary-value problem (1)–(2) with one straight angle, i.e., to a problem with twice as many equations and boundary conditions; moreover, proper ellipticity and the covering condition are preserved.

Consider the auxiliary problem

$$L(0, 0, \partial/\partial x, \partial/\partial y)U = f, \quad (x, y) \in \Omega_0; \quad (10)$$

$$B(0, 0, \partial/\partial x, \partial/\partial y)U = \varphi, \quad (x, y) \in \Gamma_0; \quad (11)$$

$$C^+(0, 0, \partial/\partial x, \partial/\partial y)U^+ - C^-(0, 0, \partial/\partial x, \partial/\partial y)U^- = \psi, \quad (x, y) \in \gamma_0, \quad (12)$$

where Ω_0, Γ_0 are the same as in problem (4)–(5), and γ_0 is the tangent to γ . Passing to the “doubled” problem, we construct $R(\lambda)$.

Theorem 3. If $R(\lambda)$ of the “doubled” problem (10)–(12) has no poles on the line $\text{Im } \lambda = h$, then for $(f, \varphi, \psi) \in \dot{H}_{2(k-h-1)}^k$ there exists a unique solution of problem (10)–(12) satisfying the estimate

$$\|U_j\|_{\dot{W}_2^{k+t_j}_{2(k-h-1)}} \leq C \|f, \varphi, \psi\|_{\dot{H}_{2(k-h-1)}^k}. \quad (13)$$

If the domain Ω is divided by two intersecting curves γ_1 and γ_2 , which do not have a common tangent at the point of intersection, then, after performing the transformations $x_1 = y - y_{\gamma_1}$, $y_1 = y - y_{\gamma_2}$, by analogous reasoning we obtain a

“quadrupled” boundary-value problem with one corner. Consider the auxiliary problem

$$L(0, 0, \partial/\partial x, \partial/\partial y)U = f, \quad (x, y) \in \Omega_0; \quad (14)$$

$$C_\nu^+(0, 0, \partial/\partial x, \partial/\partial y)U^+ - C_\nu^-(0, 0, \partial/\partial x, \partial/\partial y)U^- = \varphi_\nu, \quad (x, y) \in \gamma_{\nu 0}, \quad \nu = 1, 2. \quad (15)$$

Theorem 4. If $R(\lambda)$ of the “quadrupled” problem (14)–(15) has no poles on the line $\text{Im } \lambda = h$, then for $(f, \varphi) \in \dot{H}_{2(k-h-1)}^k$ there exists a unique solution of problem (14)–(15) satisfying the estimate

$$\|U_j\|_{\dot{W}_{2(k-h-1)}^{k+t_j}} \leq C \|f, \varphi\|_{\dot{H}_{2(k-h-1)}^k}.$$

Theorems 1, 3, and 4 make it possible to construct a regularizer for problem (1)–(3) (see ^(1, 4, 5)); therefore the following is true.

Theorem 5. If R_ν of the auxiliary problems (4)–(5) for each corner point, R_μ of the “doubled” auxiliary problems for each point of intersection of Γ, γ , and $R_{\mu\nu}$ of the “quadrupled” auxiliary problems for each point of intersection of the curves (γ_μ, γ_ν) have no poles on one and the same line $\text{Im } \lambda = h$, then for $(f, \varphi, \psi) \in \dot{H}_{2(k-h-1)}^k$, upon satisfaction of a finite number of conditions of the type of functionals in $\dot{H}_{2(k-h-1)}^k$, there exists a solution of problem (1)–(3), $U_j \in \dot{W}_{2(k-h-1)}^{k+t_j}$, $j = 1, \dots, N$. If it is unique, then an estimate of type (13) is valid.

Remark. The curves γ_ν may close up without reaching the boundary. If they do not intersect, then normal solvability holds (see ⁽³⁾).

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REFERENCES

1. V. A. Kondrat’ev, *Tr. Mosk. matem. obshch.*, **16**, 209 (1967).
2. S. Agmon, A. Douglis, L. Nirenberg, *Comm. Pure and Appl. Math.*, **17**, 35 (1964).

3. N. V. Zhitarashu, *DAN*, **165**, No. 1, 24 (1965).
4. M. S. Agranovich, M. I. Vishik, *UMN*, **20**, no. 3, 53 (1964).
5. V. A. Solonnikov, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **92**, 233 (1966).

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