

# ON A SINGULAR NICOLETTI PROBLEM

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON A SINGULAR NICOLETTI PROBLEM

*(Presented by Academician I. N. Vekua on 11 XI 1968)*

In the present note we consider the boundary-value problem

$$dx_i/dt = f_i(t, x_1, \dots, x_n) \quad (i = 1, 2, \dots, n); \quad (1)$$

$$x_i(t_i) = 0 \quad (i = 1, 2, \dots, n), \quad (2)$$

where  $f_i(t, x_1, \dots, x_n)$  ( $i = 1, 2, \dots, n$ ) are real functions defined in the domain  $R_{a-b}(a \leq t \leq b, -\infty < x_1, \dots, x_n < +\infty)$ ,  $-\infty < a < b < +\infty$ , and  $t_i \in [a, b]$  ( $i = 1, 2, \dots, n$ ). By a solution of this problem we mean absolutely continuous functions  $x_1(t), \dots, x_n(t)$  on  $[a, b]$  satisfying the system (1) for almost all  $t \in [a, b]$  and the boundary conditions (2).

Problem (1)–(2) was posed by O. Nicoletti <sup>(4)</sup>. Various sufficient conditions for the existence and uniqueness of its solution were given in <sup>(1–7)</sup>. In contrast to the cases considered by other authors, in the present note it is assumed that the functions  $f_i(t, x_1, \dots, x_n)$  ( $i = 1, 2, \dots, n$ ), having singularities at  $t = t_i$  ( $i = 1, 2, \dots, n$ ), in general are not summable with respect to  $t$  on the interval  $[a, b]$ .

Below we adopt the following notation:

The notation  $\varphi(t, x_1, \dots, x_m) \in K(a, b)$ , where  $m$  is an arbitrary natural number, means that the function  $\varphi(t, x_1, \dots, x_m)$ , for every  $r \in (0, +\infty)$ , satisfies the Carathéodory conditions in the domain  $a \leq t \leq b$ ,  $-r \leq x_1, \dots, x_m \leq r$ , i.e.,  $\varphi(t, x_1, \dots, x_m)$  is measurable in  $t$  on the interval  $[a, b]$  for all  $x_i \in [-r, r]$  ( $i = 1, 2, \dots, m$ ), is continuous in  $x_1, \dots, x_m$  in the domain  $-r \leq x_1, \dots, x_m \leq r$  for almost all  $t \in [a, b]$ , and

$$\sup_{|x_i| \leq r \ (i=1,2,\dots,m)} |\varphi(t, x_1, \dots, x_m)| \in L(a, b).$$

By  $\overline{K}(a, b; t_0)$ , where  $t_0 \in [a, b]$ , is denoted the set of all functions belonging to the set  $K(\alpha, \beta)$  for all  $\alpha$  and  $\beta$  satisfying  $a \leq \alpha < \beta \leq b$ ,  $t_0 \in [\alpha, \beta]$ .

$$l_k = (k-1)^{1/k} \left( \frac{k}{\pi} \sin \frac{\pi}{k} \right)^{-1}.$$

**Theorem 1.** *Let*

$$f_i(t, x_1, \dots, x_n) \in \overline{K}(a, b; t_i) \quad (i = 1, 2, \dots, n) \quad (3)$$

*and let the inequalities*

$$f_i(t, x_1, \dots, x_n) \operatorname{sign}[(t - t_i)x_i] \leq g_i(t, |x_1|, \dots, |x_n|) \quad (i = 1, 2, \dots, n),$$

*hold in the domain  $R_{ab}$ , where the functions  $g_i(t, x_1, \dots, x_n) \in K(a, b)$  ( $i = 1, 2, \dots, n$ ) are nonnegative and nondecreasing in each  $x_j$ ,  $1 \leq j \leq n$ . Moreover, suppose that there exists a positive number  $M$  such that*

$$\rho_i(t) \leq M \quad \text{for } a \leq t \leq b \quad (i = 1, 2, \dots, n)$$

*whatever absolutely continuous functions  $\rho_i(t)$  ( $i = 1, 2, \dots, n$ ) on the interval  $[a, b]$  may be that satisfy on this interval the conditions*

$$\rho_i(t_i) = 0, \quad 0 \leq \rho'_i(t) \operatorname{sign}(t - t_i) \leq g_i(t, \rho_1(t), \dots, \rho_n(t))$$

$$(i = 1, 2, \dots, n). \quad (4)$$

Then problem (1)–(2) has at least one solution.

**Corollary 1.** Suppose that conditions (3) are satisfied and, in the domain  $R_{ab}$ , the inequalities

$$\begin{aligned} f_i(t, x_1, \dots, x_n) \operatorname{sign}[(t - t_i)x_i] &\leq \\ &\leq h_i(t, |x_1|, \dots, |x_{i-1}|)\omega_i(|x_i|) \quad (i = 1, 2, \dots, n), \end{aligned}$$

hold, where  $h_1(t, x_0) \equiv h_1(t) \in L(a, b)$ ,  $h_i(t, x_1, \dots, x_{i-1}) \in K(a, b)$  ( $i = 2, \dots, n$ ), and  $\omega_i(x)$  ( $i = 1, 2, \dots, n$ ) are continuous, positive, nondecreasing functions on the interval  $[0, +\infty)$  satisfying the conditions

$$\int_0^{+\infty} \frac{dt}{\omega_i(t)} = +\infty \quad (i = 1, 2, \dots, n).$$

Then problem (1)–(2) has at least one solution.

**Corollary 2.** Suppose that conditions (3) are satisfied and, in the domain  $R_{ab}$ , the inequalities

$$f_i(t, x_1, \dots, x_n) \operatorname{sign}[(t - t_i)x_i] \leq$$

$$\leq g_i \left( t, \sum_{j=1}^n |x_j| \right) |x_i| + h_i \left( t, \sum_{j=1}^n |x_j| \right) \quad (i = 1, 2, \dots, n),$$

hold; the functions  $g_i(t, x) \in K(a, b)$  and  $h_i(t, x) \in K(a, b)$  ( $i = 1, 2, \dots, n$ ) are nonnegative and nondecreasing in  $x$ , and

$$\max_{a \leq t \leq b} \sum_{i=1}^n \left| \int_{t_i}^t \left( h_i(\tau, x) \exp \left\{ \left| \int_{\tau}^t g_i(s, x) ds \right| \right\} \right) d\tau \right| < x \quad \text{for } x > M,$$

where  $M$  is some positive constant. Then problem (1)–(2) has at least one solution.

**Corollary 3.** Suppose that conditions (3) are satisfied and, in the domain  $R_{ab}$ , the inequalities

$$f_i(t, x_1, \dots, x_n) \operatorname{sign}[(t - t_i)x_i] \leq g_i \left( t, \sum_{j=1}^n |x_j|^k \right) \quad (i = 1, 2, \dots, n), \quad (5)$$

hold,  $1 < k < +\infty$ , and the functions  $g_i(t, x) \in K(a, b)$  ( $i = 1, 2, \dots, n$ ) are nonnegative, nondecreasing in  $x$ , and

$$\int_a^b g(t, x) dt < l_k \quad \text{for } x > M, \quad (6)$$

where  $M$  is some positive constant, and

$$g(t, x) = \sup_{x \exp(-l_k) \leq y \leq x} \left\{ \sum_{i=1}^n \left[ \frac{g_i(t, y^k)}{y} \right]^k \right\}^{1/k}.$$

Then problem (1)–(2) has at least one solution.

Condition (6) is satisfied, for example, if

$$\left\{ \sum_{i=1}^n [g_i(t, |x|^k)]^k \right\}^{1/k} \leq g(t)|x| + h(t, |x|),$$

$$g(t) \geq 0 \quad \text{for } a \leq t \leq b, \quad \int_a^b g(t) dt < l_k,$$

and the function  $h(t, x) \in K(a, b)$  is nonnegative, nondecreasing in  $x$ , and

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \int_a^b h(t, x) dt = 0.$$

If, in addition, one assumes that  $f_i(t, x_1, \dots, x_n) \in K(a, b)$  ( $i = 1, 2, \dots, n$ ),

$k = 2$  and, instead of (5), the inequalities

$$|f_i(t, x_1, \dots, x_n)| \leq g_i \left( t, \sum_{j=1}^n |x_j|^2 \right) \quad (i = 1, 2, \dots, n),$$

hold, then we obtain the theorem of Lasota–Olech <sup>(2)</sup>.

**Theorem 2.** Suppose that for  $(t, x_1, \dots, x_n), (t, y_1, \dots, y_n) \in R_{ab}$  the inequalities

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)] \operatorname{sign}[(t - t_i)(x_i - y_i)] &\leq \\ &\leq g_i(t, |x_1 - y_1|, \dots, |x_n - y_n|) \quad (i = 1, 2, \dots, n), \end{aligned}$$

hold, where the functions  $g_i(t, x_1, \dots, x_n) \in K(a, b)$  ( $i = 1, 2, \dots, n$ ) are nonnegative and nondecreasing in each  $x_j$ ,  $1 \leq j \leq n$ . Moreover, there do not exist absolutely continuous functions  $\rho_i(t)$  ( $i = 1, 2, \dots, n$ ) on the interval  $[a, b]$ , among which at least one is not identically equal to zero and which would satisfy conditions (4). Then problem (1)–(2) has at most one solution.

**Corollary 1.** Suppose that for  $(t, x_1, \dots, x_n), (t, y_1, \dots, y_n) \in R_{ab}$  the inequalities

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)] \operatorname{sign}[(t - t_i)(x_i - y_i)] &\leq \\ &\leq g_i(t) |x_i - y_i| + h_i(t) \sum_{j=1}^n |x_j - y_j| \quad (i = 1, 2, \dots, n), \end{aligned}$$

hold, where the functions  $g_i(t) \in L(a, b)$  and  $h_i(t) \in L(a, b)$  ( $i = 1, 2, \dots, n$ ) are nonnegative and

$$\max_{a \leq t \leq b} \sum_{i=1}^n \left| \int_{t_i}^t g_i(\tau) \exp \left( \int_{\tau}^t h_i(s) ds \right) d\tau \right| < 1.$$

Then problem (1)–(2) has at most one solution.

**Corollary 2.** Suppose that for  $(t, x_1, \dots, x_n), (t, y_1, \dots, y_n) \in R_{ab}$  the inequalities

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)] \operatorname{sign}[(t - t_i)(x_i - y_i)] &\leq \\ &\leq g_i(t) \left\{ \sum_{j=1}^n |x_j - y_j|^k \right\}^{1/k} \quad (i = 1, 2, \dots, n), \end{aligned}$$

hold, where  $1 < k < +\infty$ , and the functions  $g_i(t) \in L(a, b)$  ( $i = 1, 2, \dots, n$ ) are nonnegative and

$$\int_a^b \left\{ \sum_{i=1}^n [g_i(t)]^k \right\}^{1/k} dt < l_k.$$

Then problem (1)–(2) has at most one solution.

According to Corollaries 2 and 3 of Theorem 1, it is easy to conclude that if  $f(t, 0, \dots, 0) \in L(a, b)$ ,  $f_i(t, x_1, \dots, x_n) \in K(a, b; t_i)$  ( $i = 1, 2, \dots, n$ ) and either the conditions of Corollary 1 or the conditions of Corollary 2 of Theorem 2 are satisfied, then problem (1)–(2) has one and only one solution.

**Theorem 3.** Suppose that

$$f_i(t, |t - t_1|x_1, \dots, |t - t_n|x_n) \in K(a, b; t_i),$$

$$\text{vrai max}_{a \leq t \leq b} |f_i(t, 0, \dots, 0)| < +\infty \quad (i = 1, 2, \dots, n)$$

and, for  $(t, x_1, \dots, x_n), (t, y_1, \dots, y_n) \in R_{ab}$ , the inequalities

$$\begin{aligned} & [f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)] \text{sign}[(t - t_i)(x_i - y_i)] \leq \\ & \leq \alpha_i \left| \frac{x_i - y_i}{t - t_i} \right| + \beta_i \sum_{j=1}^n \left| \frac{x_j - y_j}{t - t_j} \right| \quad (i = 1, 2, \dots, n), \end{aligned}$$

where  $0 \leq \alpha_i, \beta_i < 1$  ( $i = 1, 2, \dots, n$ ) and

$$u \sum_{i=1}^n \frac{\beta_i}{1 - \alpha_i} < 1.$$

Then system (1) has one and only one solution  $x_1(t), \dots, x_n(t)$ , satisfying the boundary conditions

$$\overline{\lim}_{t \rightarrow t_i} \left| \frac{x_i(t)}{t - t_i} \right| < +\infty \quad (i = 1, 2, \dots, n).$$

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