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Abstract

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MATHEMATICS

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ON CANONICAL DECOMPOSITIONS OF EQUATIONS OF A LOCAL ANALYTIC QUASIGROUP

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1. Let X, Y, Z be three analytic manifolds of the same dimension r . An analytic mapping $f : X \times Y \rightarrow Z$ is called a local analytic quasigroup Q_r if, for $a \in X$, $b \in Y$, and $c = f(a, b) \in Z$, the following conditions are satisfied:
 - a) For every neighborhood U_c of the point c there exist neighborhoods U_a and U_b of the points a and b such that, for any $x \in U_a$, $y \in U_b$, the function $f(x, y)$ is defined and $f(x, y) \in U_c$.
 - b) For every neighborhood U_a of the point a there exist neighborhoods U_b and U_c of the points b and c such that, for any $y \in U_b$ and $z \in U_c$, the equation $f(x, y) = z$ has a unique solution with respect to x , and $x \in U_a$.
 - c) For every neighborhood U_b of the point b there exist neighborhoods U_a and U_c of the points a and c such that, for any $x \in U_a$ and $z \in U_c$, the equation $f(x, y) = z$ has a unique solution with respect to y , and $y \in U_b$.

If the manifolds X, Y, Z coincide, $X \equiv Y \equiv Z$, and there exists an element e such that $f(x, e) = f(e, x) = x$, then the quasigroup Q_r becomes a local analytic loop ⁽¹⁾. If, moreover, the associativity condition $f[x, f(y, z)] = f[f(x, y), z]$ is satisfied on the quasigroup Q_r , then it becomes a local r -parameter Lie group.

In the present paper canonical decompositions are obtained for the analytic function $z = f(x, y)$ defining the quasigroup Q_r , which pass into the Campbell-Hausdorff formula ⁽²⁾ when the quasigroup Q_r becomes a local Lie group. The constructed canonical decompositions are applied to the study of certain special classes of local analytic quasigroups.

2. Let a and b be arbitrary points of the manifolds X and Y , and let $c = f(a, b)$ be a point belonging to the manifold Z . Introduce local coordinates in neighborhoods U_a, U_b , and U_c of the points a, b , and c . Then in these neighborhoods the mapping $f : X \times Y \rightarrow Z$ is written in the form:

$$z^i = f^i(x^j, y^k), \quad i, j, k = 1, \dots, r, \quad (1)$$

where x^i are the coordinates of the point $x \in U_a$, y^i are the coordinates of the point $y \in U_b$, z^i are the coordinates of the point $z \in U_c$, and f^i are analytic functions.

Suppose that the points a and b of the manifolds X and Y are determined by the coordinates $x^i = 0$ and $y^i = 0$. Then in the neighborhoods U_a and U_b the right-hand sides of equations (1) can be represented in the form of convergent power series:

$$z^i = \sum_{s=0}^{\infty} \Lambda_{(s)}^i(x^j, y^k), \quad (2)$$

where

$$\Lambda_{(s)}^i(x^j, y^k) = \frac{1}{s!} \sum_{p=0}^s C_s^p \lambda_{j_1 \dots j_p r+j_{p+1} \dots r+j_s}^i x^{j_1} \dots x^{j_p} y^{j_{p+1}} \dots y^{j_s}. \quad (3)$$

homogeneous polynomials of degree s in x^j, y^k . Their coefficients $\lambda_{j_1 \dots j_p r+j_{p+1} \dots r+j_s}^i$ are symmetric in the indices $j_1 \dots j_p$ and $j_{p+1} \dots j_s$, and C_s^p are binomial coefficients.

The local coordinates of points of the neighborhoods U_a, U_b , and U_c of the analytic manifolds X, Y, Z admit transformations of the form

$$\tilde{x}^i = \varphi^i(x^j), \quad \tilde{y}^i = \psi^i(y^j), \quad \tilde{z}^i = \chi^i(z^j), \quad (4)$$

where $\varphi^i, \psi^i, \chi^i$ are uniquely invertible analytic functions such that $\varphi^i(0) = 0$, $\psi^i(0) = 0$. Such transformations are called isotopic coordinate transformations of the quasigroup Q_r . These transformations make it possible to reduce the expansions (2) to a certain simplest form, namely:

Theorem 1. *With the aid of transformations (4), the polynomials $\Lambda_{(s)}^i(x^j, y^k)$ occurring in the expansions (2) can be reduced to the following canonical form:*

$$\begin{aligned} \Lambda_{(0)}^i &= 0, & \Lambda_{(1)}^i &= x^i + y^i, & \Lambda_{(2)}^i &= a_{jk}^i x^j y^k, \\ \Lambda_{(s)}^i &= \frac{1}{s!} \sum_{p=1}^{s-1} C_s^p a_{j_1 \dots j_p r+j_{p+1} \dots r+j_s}^i x^{j_1} \dots x^{j_p} y^{j_{p+1}} \dots y^{j_s}, \end{aligned} \quad (5)$$

where the coefficients of these polynomials satisfy the relations

$$a_{(jk)}^i = 0, \quad b_{(j_1 \dots j_s)}^i = 0, \quad (6)$$

where

$$b_{(j_1 \dots j_s)}^i = \sum_{p=1}^{s-1} C_s^p \sigma^{s-p} a_{j_1 \dots j_p r+j_{p+1} \dots r+j_s}^i,$$

and σ is some fixed real number, $\sigma \neq 0, -1$.

Expansions of equations (2) satisfying the conditions of this theorem will be called canonical expansions of the equations of the quasigroup Q_r . Thus, the quasigroup Q_r has not one but an entire pencil of canonical expansions, depending on the parameter σ . The variables x^i and y^i occurring in the canonical expansions of the quasigroup Q_r will be called its canonical parameters.

The uniqueness of the canonical expansions is confirmed by the following theorem:

Theorem 2. *The canonical expansion of the equations of a local analytic quasigroup preserves its form if and only if the variables entering into it are transformed according to the formulas*

$$\tilde{x}^i = \alpha_j^i x^j, \quad \tilde{y}^i = \alpha_j^i y^j, \quad \tilde{z}^i = \alpha_j^i z^j. \quad (7)$$

In this case the coefficients a_{jk}^i and $a_{j_1 \dots j_p r+j_{p+1} \dots r+j_s}^i$ of the canonical expansion are transformed according to the tensor law.

The proof of Theorems 1 and 2 is carried out by the method of complete induction on the degree s of the polynomials $\Lambda_{(s)}(x, y)$.

3. Let us identify the points of the neighborhoods U_a, U_b , and U_c that have identical coordinates, and denote by the letter e the point obtained by identifying the points a, b, c . Then the local quasigroup Q_r will be defined in some neighborhood U of this point, and the binary quasigroup operation will be an analytic mapping $f: U \times U \rightarrow U$. In this case the point e will correspond to the identity element of the quasigroup Q_r , since by virtue of relations (5)

$$f(x, e) = f(e, x) = x.$$

Consequently, the quasigroup Q_r becomes a local analytic loop. Moreover, by virtue of relations (5) and (6),

$$f(x, \sigma x) = (1 + \sigma)x, \quad (8)$$

where the latter relation is satisfied for one fixed value of the parameter $\sigma \neq 0, -1$. We shall call condition (8) the normalization condition for the equations of the quasigroup.

4. Let Q_1 be a one-parameter quasigroup. By virtue of the first of conditions (6), its canonical decompositions are written in the form

$$z = x + y + \sum_{s=3}^{\infty} \Lambda_{(s)}(x, y).$$

If we introduce the notation

$$\underbrace{a_{1\dots 1}}_p \underbrace{2\dots 2}_q = a_{pq},$$

then the polynomials $\Lambda_{(s)}(x, y)$, for $s \geq 3$, take the form

$$\Lambda_{(s)}(x, y) = \frac{1}{s!} \sum_{p=1}^{s-1} C_s^p a_{p, s-p} x^p y^{s-p}, \quad (9)$$

where

$$\sum_{p=1}^{s-1} C_s^p \sigma^{s-p} a_{p, s-p} = 0.$$

A direct verification shows that if the quasigroup Q_1 becomes a one-parameter Lie group G_1 , then all its canonical decompositions take the form

$$z = x + y.$$

Consequently, if the quasigroup Q_r becomes a Lie group, then the canonical parameters introduced on Q_r pass into the ordinary canonical parameters of the local Lie group G_r . All canonical decompositions of the quasigroup Q_r then pass into the Campbell–Hausdorff formula for the Lie group G_r .

5. Consider on the quasigroup Q_r a one-parameter subquasigroup Q_1 . It is defined on Q_r by the equations:

$$x^i = x^i(u), \quad y^i = y^i(v), \quad f^i[x^j(u), y^h(v)] = z^i(w), \quad (10)$$

where $w = \varphi(u, v)$. From these relations the following theorem follows easily:

Theorem 3. In order that the first two of equations (10) define on the quasigroup Q_r a one-parameter subquasigroup Q_1 , it is necessary and sufficient that the relations

$$\lambda \frac{\partial z^i}{\partial x^j} \frac{dx^j}{du} + \mu \frac{\partial z^i}{\partial y^j} \frac{dy^j}{dv} = 0, \quad (11)$$

hold, where λ and μ are analytic functions of u and v .

Relations (11) are differential equations of one-parameter subquasigroups on Q_r . These relations make it possible to prove the following theorem:

Theorem 4. In order that on the loop Q_r , for each direction issuing from the point e , there exist a one-parameter subloop Q_1 tangent to this direction, it is necessary and sufficient that the coefficients of at least one of its canonical decompositions satisfy the relations:

$$b_{p(j_1 \dots j_s)}^i = \delta_{(j_1 p_{j_2 \dots j_s})}^i b_{p_{j_2 \dots j_s}}^i, \quad (12)$$

where

$$b_{p(j_1 \dots j_s)}^i = a_{j_1 \dots j_p}^i r + j_{p+1} \dots r + j_s.$$

We note that if the condition of Theorem 4 is fulfilled, the coefficients of any of the canonical decompositions of the quasigroup Q_r will satisfy relations (12).

The three-web ⁽³⁾ corresponding to the quasigroup Q_r defined by Theorem 4 was considered by us in ⁽⁴⁾ under the name of a transversal-geodesic web.

6. The following theorem answers the question of when the one-parameter subloops Q_1 of the loop Q_r become groups.

Theorem 5. *In order that on the loop Q_r , for every direction issuing from the point e , there exist a one-parameter subgroup G_1 tangent to this direction, it is necessary and sufficient that the coefficients of at least one of its canonical decompositions satisfy the conditions*

$$b_{p(j_1 \dots j_s)}^i = 0. \quad (13)$$

The condition contained in Theorem 5 is equivalent to the monoassociativity ⁽³⁾ of the loop Q_r . Therefore condition (13) is a necessary and sufficient condition for the monoassociativity of the loop Q_r . Note that in ⁽¹⁾ the existence of one-parameter subgroups of a local loop Q_r was proved only for alternative loops, which form a narrower class than monoassociative loops.

When condition (13) is fulfilled, for arbitrary x belonging to a sufficiently small neighborhood of the point e of the loop Q_r , and for sufficiently small real λ and μ , the relation

$$f(\lambda x, \mu x) = (\lambda + \mu)x$$

will hold, and this means that the normalization condition (8) is fulfilled on the quasigroup Q_r for every $\sigma \neq 0, -1$. Hence it follows:

Theorem 6. *In order that all the canonical decompositions of the equations of the quasigroup Q_r introduced in Theorem 1 coincide with one another, it is necessary and sufficient that this quasigroup be monoassociative.*

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CITED LITERATURE

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