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B. Efimov

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Abstract

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MATHEMATICS

B. EFIMOV

SOLUTION OF SOME PROBLEMS ON DYADIC BICOMPACTA

(Presented by Academician P. S. Aleksandrov, 20 XII 1968)

Here we shall establish a certain connection between dyadicity, dimension, and metrizable, solve a number of problems posed in the papers ^(1, 2), and give answers to questions proposed to the author at P. S. Aleksandrov's seminar at Moscow University. From the paper ⁽³⁾ we take all notation and concepts without further comment.

§ 1. **Openly dyadic bicompecta.** We shall call a bicompectum X openly dyadic if it is an open continuous image of a Tikhonov product of metric compacta

$$\prod_{\alpha \in A} R_{\alpha}.$$

The class $\mathfrak{R}$ of openly dyadic bicompecta is the smallest class closed with respect to Tikhonov products and open mappings and containing all metric compacta. Every finite-dimensional bicompect topological group is an openly dyadic bicompectum. Let us note a number of specific properties of openly dyadic bicompecta.

Let $X \in \mathfrak{R}$. Then:

(α). The closure of any open set U in X is of type G_{δ} , and the nonempty kernel of any closed set $F \subset X$ is of type F_{σ} . Moreover, this property is hereditary with respect to closed subsets of X of type G_{δ} .

(β). If X is zero-dimensional, then X is an open image of D^{τ} , where $\tau = wX$.

(γ). For any open mapping

$$f : \prod_{\alpha \in A} R_{\alpha} \rightarrow X$$

there exists an open mapping

$$g : \prod_{\alpha \in B} R_\alpha \rightarrow X,$$

where $|B| = wX$, such that $f = g\pi$, if π is the projection of

$$\prod_{\alpha \in A} R_\alpha \quad \text{onto} \quad \prod_{\alpha \in B} R_\alpha.$$

(δ). The set

$$M_n = \{x \in X, \chi(x, X) \leq n\}$$

is closed, and moreover $\chi(M_n, X) \leq n$ and $wM_n \leq n$. The set $\{x \in X, \chi(x, X) \geq n^+\}$ is open. The set $\{x \in X, \chi(x, X) = n^+\}$ is the intersection of an open set with a closed one. The set $\{x \in X, \chi(x, X) \geq n\}$, if n is the sum of a countable number of smaller cardinals, is of type G_δ .

Lemma 1. If a bicompactum X has property (α), then the boundary of every canonical closed subset X is of type G_δ . Moreover, this property is hereditary with respect to closed subsets of X of type G_δ .

Lemma 2. Let X be a locally connected bicompactum, and let F be a closed zero-dimensional subset of type G_δ in X . Then $wF \leq \aleph_0$, and for every point $x \in F$ we have $\chi(x, X) \leq \aleph_0^*$.

Theorem 1. Every finite-dimensional (in the sense of ind) locally connected openly dyadic bicompactum X is metrizable.

Proof. We shall prove that in X there exists an everywhere dense subset M with the first axiom of countability. It will follow from this that,

* Compare with the proof of Lemma 4 in (⁴). By $\chi(F, X)$ we denote the character of $F \subset X$, and by wX the weight of X .

by virtue of one theorem of the author, the metrizability of X (see (2), p. 245, or (5), pp. 162–163). Let $\text{ind } X = n$. Without loss of generality we shall assume that X is connected. Otherwise X splits into a finite number of components. For every connected locally connected neighborhood $V \subset X$ there exists a neighborhood U such that $\overline{U} \subset V$ and $\text{ind}(\text{Fr } U) \leq n - 1$. Put $L_1 = \overline{U} \setminus \text{int } \overline{U}$. Since $L_1 \subset \text{Fr } U$, and by the monotonicity of the dimension ind for regular spaces ((5), p. 264), we have $\text{ind } L_1 \leq n - 1$. Since X is connected, $L_1 \neq \emptyset$. Applying Lemma 1, we obtain that $\chi(L_1, X) \leq \aleph_0$. Let $\text{ind } L_1 = k_1$. Clearly $k_1 \leq n - 1$. If $k_1 = 0$, then L_1 is a closed zero-dimensional subset of type G_δ lying in a locally connected bicompactum. Hence by Lemma 2 for each point $x \in L_1$ we have $\chi(x, X) \leq \aleph_0$, as required. Let $k_1 \geq 1$. We shall continue the reasoning by induction. Suppose that a nonempty closed $L_i \subset V$ of type G_δ in X has been

found, with $0 \leq \text{ind } L_i \leq n-i$. Let $\text{ind } L_i = k_i$. Clearly $0 \leq k_i \leq n-i$. If $k_i = 0$, then, as we showed above, our goal has been attained. If $k_i \geq 1$, then there exists an open $W \subset L_i$ such that $\text{ind}(\text{Fr } W) \leq k_i - 1$. Put $L_{i+1} = \overline{W} \setminus \text{int } \overline{W}$. Clearly $L_{i+1} \subset \text{Fr } W \subset L_i \subset V$ and $\text{ind } L_{i+1} \leq n-i-1$. Since, by assumption, $\chi(L_i, X) \leq \aleph_0$, by Lemma 1 L_{i+1} is of type G_δ in X . The induction is complete. Thus, in no more than n steps we shall find a nonempty L_n of type G_δ in X , with $\text{ind } L_n = 0$ and $L_n \subset V$. Hence by Lemma 2 there follows the existence of a point $x \in V$ such that $\chi(x, X) \leq \aleph_0$. In view of the arbitrary choice of V , we obtain the required set M . The theorem is proved.

Theorem 2. *Every connected openly dyadic peripherally metrizable bicom pactum X is metrizable.**

Proof. Again we prove that in X there exists an everywhere dense subset M satisfying the first axiom of countability. Let $\mathfrak{U} = \{U\}$ be a base of open sets in X such that for every $U \in \mathfrak{U}$ we have $w(\text{Fr } U) \leq \aleph_0$. Let V be an arbitrary open set in X , and let $U \in \mathfrak{U}$ be such that $\overline{U} \subset V$. Put $L = \overline{U} \setminus \text{int } \overline{U}$. Then, by Lemma 1, $\chi(L, X) \leq \aleph_0$. On the other hand, $L \subseteq \text{Fr } U$, and consequently $wL \leq \aleph_0$. Since X is connected, $L \neq \emptyset$. Hence for every point $x \in L$ we have

$$\chi(x, X) \leq \chi(x, L) \cdot \chi(L, X) \leq wL \cdot \chi(L, X) \leq \aleph_0 \cdot \aleph_0 = \aleph_0.$$

Since $L \subset V$, we have $x \in V$. Since V is an arbitrary open set, the set M of all points of countable character is dense in X . The theorem is proved.

Let us note that in proving Theorems 1 and 2 we used only dyadicity and property (α) . Therefore the following problem naturally arises:

Is every dyadic bicom pactum satisfying condition (α) openly dyadic?

Let us further note that Theorem 2 admits the following natural generalization.

Theorem 3. *If in a connected openly dyadic bicom pactum X there exists a π -base $\mathfrak{U}$ such that for every $U \in \mathfrak{U}$ it follows that $w(\text{Fr } U) \leq \tau$, then $wX \leq \tau$.*

§ 2. Wild dyadic bicom pacta

A dyadic bicom pactum X of weight τ will be called **wild** if: a) for every open $U \subset X$ we have $wU = wX = \tau$; b) for every open $U \subset X$ there exists a point $x \in U$ such that $\chi(x, X) < \tau$.

Let us note that, as follows from one theorem of the author ((2), p. 231), the weight of any wild dyadic bicom pactum is a limit cardinal number. On the other hand, in (2), p. 234, it is shown that under the assumption (GCH)** there does not exist a wild dyadic bicom pactum whose weight is an inaccessible number. However, the following is true

* This theorem gives a partial answer to a question of V. V. Proizvolov.

** (GCH) —generalized continuum hypothesis, (CH) —continuum hypothesis.

Theorem 4. For every singular cardinal number τ there exists a wild dyadic bicom pactum $S(\tau)$ of weight τ which, first, is an open image of D^τ and, second, an irreducible image of D^τ .*

We shall indicate only the construction of the bicom pactum $S(\tau)$. Denote by $\mathfrak{D}_\tau = D^\tau \oplus (x)$ the topological sum ⁽⁵⁾ of D^τ and an isolated point (x) ; by $\mathfrak{F}_\tau$ the Aleksandrov compactification of the topological sum of a countable number of copies of D^τ ; by $\mathfrak{L}_\tau$ the space obtained from D^τ by contracting to a point some closed nowhere dense subset of type G_δ .

Lemma 3. If $\tau \geq \aleph_1$, then the spaces $(\mathfrak{D}_\tau)^{\aleph_0}$, $\mathfrak{F}_\tau$, and $\mathfrak{L}_\tau$ are homeomorphic to one another.

Lemma 4. The space $\mathfrak{D}_\tau$ is an open image of D^τ , and the space $\mathfrak{L}_\tau$ is an irreducible image of D^τ .

Let $\tau = \sum_{\alpha \in A} \mathfrak{n}_\alpha$, $\mathfrak{n}_\alpha < \tau$, $|A| < \tau$ be a singular number. Then

$$S(\tau) = \prod_{\alpha \in A} \mathfrak{F}_{\mathfrak{n}_\alpha}$$

is the required wild dyadic bicom pactum of weight τ .

§ 3. Dyadicity and hypotheses of set theory. Let X be a topological space, $M \subset X$, and $x \in X$. We shall call the local power $\mathfrak{t}(x, M)$ of the set M relative to the point $x \in X$ the number $\mathfrak{t}(x, M) = \min |Ox \cap M|^{**}$, if $Ox \in \mathfrak{F}$, where $\mathfrak{F}$ is the filter of all neighborhoods of the point x in X . A point $x \in X$ is called a point of extremal accumulation ⁽¹⁾ for the set M if $\mathfrak{t}(x, M) > \mathfrak{t}(y, M)$ for all $y \in X$ and $y \neq x$. In other words, a point $x \in X$ is a point of extremal accumulation for the set M if the function $\mathfrak{t}(x, M)$, defined on X , has an absolute maximum at the point x . The space X is called an χ -space ⁽¹⁾ if for every $M \subset X$ and every point $x \in \overline{M}$ there exists $M' \subset M$ for which x is a point of extremal accumulation in X .

An easy consequence of Theorem 9 from the author' s paper ⁽³⁾ is the following

Theorem 5 (CH). Every nonmetrizable dyadic bicom pactum topologically contains the Stone-Čech compactification βN of the natural numbers.***

Since βN is not an χ -space, while every closed subspace of an χ -space is an χ -space, we immediately obtain the following theorem, which is an answer to a question of A. V. Arhangel' skii and V. I. Ponomarev ⁽¹⁾.

Theorem 6 (CH). Every dyadic bicom pactum which is an χ -space is metrizable.

One says that a space X satisfies Shanin's condition if every decreasing sequence, well ordered by the inclusion relation, of nonempty open subsets of X whose intersection is empty contains a countable cofinal subsequence. The following theorem is of interest in connection with the problem posed in ⁽¹⁾, p. 995. Let us note that we do not require Shanin's condition to hold hereditarily.

Theorem 7 (CH). Every Hausdorff space X with the first axiom of countability, satisfying Shanin's condition, is separable.

Proof. Since Shanin's condition implies Suslin's condition, X satisfies the first axiom of countability and Suslin's condition. Hence, by a theorem of I. Juhász and A. Hajnal ⁽⁶⁾, it follows that $|X| \leq \exp \aleph_0$. If $|X| \leq \aleph_0$, then all is proved. If $|X| = \aleph_1$, let us enumerate the points of X by all ordinals $\leq \omega_1$. Thus, let

$$X = (x_1, x_2, \dots, x_\alpha, \dots, \alpha < \omega_1).$$

* This theorem is a solution of the problem posed in ⁽²⁾, p. 235.

** $|A| = \text{card } A$.

*** Let us note that in fact Theorem 5 is equivalent to the continuum hypothesis.

Put $F_\alpha = \{x_\beta, \beta < \alpha\}$, $\varphi_\alpha = \overline{F_\alpha}$. Suppose that X is not separable. Then for all $\alpha < \omega_1$ we have $U_\alpha = X \setminus \Phi_\alpha \neq \emptyset$. Hence the sequence $U_1 \supset U_2 \supset \dots \supset U_\alpha \supset \dots$, $\alpha < \omega_1$, is a decreasing well-ordered sequence of nonempty open sets, and

$$\bigcap_{\alpha < \omega_1} U_\alpha = \bigcap_{\alpha < \omega_1} (X \setminus \Phi_\alpha) = X \setminus \bigcup_{\alpha < \omega_1} \Phi_\alpha = X \setminus X = \emptyset.$$

It is easy to show that this sequence contains no countable cofinal subsequence, which contradicts Shanin's condition. The theorem is proved.

Corollary (CH). The Suslin continuum does not satisfy Shanin's condition.

The following theorem is a solution of the problem posed in ⁽²⁾, p. 237.

Theorem 8 (GCH). The cardinality of a dyadic bicomactum is either a power of two or is the sum of a countable number of smaller cardinals.

Theorem 9. The following two conditions are equivalent:

- a) (X is dyadic and $|X| \leq \exp \aleph_0$) $\Rightarrow$ (X is metrizable);
- b) $\exp \aleph_0 < \exp \aleph_1$.

The following theorem is a strengthening of a theorem of A. S. Esenin-Vol'pin ⁽⁷⁾.

Theorem 10. If the weight of a dyadic bicomactum X has uncountable cofinal character, then there exists in X a subset M such that $\text{int } \overline{M} \neq \emptyset$ and for every point $x \in M$ we have $\chi(x, X) = wX$.

The following theorem gives an answer to a question posed to the author by M. Choban in connection with the work ⁽⁸⁾.

Theorem 11. *Every generalized metrizable dyadic bicompactum is metrizable.*

Central Economic-Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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