

THE BOUNDARY PROJECTION OPERATOR

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Abstract

Full Text

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MATHEMATICS

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THE BOUNDARY PROJECTION OPERATOR

(Presented by Academician M. V. Keldysh, 19 VII 1968)

Consider the difference equation

$$\sum_{k \in K} A_k u_{n+k} = 0, \quad n \in D_0. \quad (1)$$

Here D_0 and K are certain finite sets of points $n = (n_1, n_2, \dots, n_s)$ and $k = (k_1, k_2, \dots, k_s)$ with integer coordinates in a space of finite dimension s ; the unknown vector-function $u = \{u_n\}$ is defined on the set D , which is swept out by the point $n+k$ when n and k independently sweep out D_0 and K , respectively; u_n is a vector of some finite dimension m ; A_k , $k \in K$, are square matrices of the same dimension, independent of n . In the grid set D we single out the subset Γ of "near-boundary points." A point r , $r \in D$, will be assigned to Γ if there exists a nonempty subset K_r of the set K such that

$$r - k \in D, \quad \text{if } k \in K_r. \quad (2)$$

With respect to the coefficients A_k of equation (1) we shall assume that, for the characteristic matrix $A(\xi)$, defined by the equality

$$A(\xi) = \sum_{k \in K} A_k \xi^{-k}, \quad \xi^k = \xi_1^{k_1} \xi_2^{k_2} \dots \xi_s^{k_s},$$

the inequality

$$\det A(\xi) \neq 0 \quad (3)$$

is satisfied for any complex $\xi_1, \xi_2, \dots, \xi_s$ satisfying the conditions

$$|\xi_1| = |\xi_2| = \dots = |\xi_s| = 1. \quad (4)$$

Requirement (3) is usually fulfilled for equations arising on the "upper" time layer when implicit difference schemes are used for solving evolution problems.

For each r , $r \in \Gamma$, introduce the matrix

$$A_r(\xi) = \sum_{k \in K_r} A_k \xi^{-k},$$

and construct the matrices β_n^r , $n \in D$, by the formulas

$$\beta_n^r = \frac{1}{(2\pi i)^s} \oint_{|\xi_s|=1} \dots \oint_{|\xi_1|=1} \frac{A^{-1}(\xi) A_r(\xi)}{\xi^{n-r+e}} d\xi_1 \dots d\xi_s,$$

where e is the s -dimensional vector each component of which is equal to one.

Consider the linear spaces U_Γ and U_D of grid m -dimensional vector-functions u_n of the argument n , $n \in \Gamma$ and $n \in D$, respectively. Define

onto U_Γ the operator P_D , $u = P_D v$, with values in U_D , by setting

$$u_n = \sum_{r \in \Gamma} \beta_n^r v_r. \quad (5)$$

Theorem 1. The lattice vector-function u , $u \in U_D$, is a certain solution of equation (1), whatever the lattice function v , $v \in U_\Gamma$.

Proof. It is enough to verify that $\beta_n^r v_r$ is a solution:

$$\sum_{k \in K} A_k \beta_{n+k}^r v_r = \frac{1}{(2\pi i)^s} \left[\oint \dots \oint \frac{A(\xi) A^{-1}(\xi) A_r(\xi)}{\xi^{n-r+e}} d\xi_1 \dots d\xi_s \right] v_r = \frac{1}{(2\pi i)^s} \left[\oint \dots \oint \sum_{k \in K_r} \frac{A_k}{\xi^{n-r+k+e}} d\xi_1 \dots d\xi_s \right] v_r$$

But the point $n = r - k$ does not belong to D_0 by virtue of (2), and at the points of D_0 equation (1) is satisfied.

Theorem 2. Let u_Γ , $u_\Gamma \in U_\Gamma$, be a lattice function formed by the boundary values of some solution u of equation (1). The solution u that coincides with u_Γ on Γ is unique and is representable in the form

$$u_n = \sum_{r \in \Gamma} \beta_n^r u_r.$$

Proof. Let u , $u = \{u_n\}$, be a solution of equation (1). Construct the generating polynomial $U(\xi)$ by setting

$$U(\xi) = \sum_{n \in D} u_n \xi^n.$$

Multiply equation (1) by ξ^n and sum over all n , $n \in D_0$. We obtain

$$0 = \sum_{n \in D} \sum_{k \in K} A_k \xi^{-k} (u_{n+k} \xi^{n+k}) = \sum_{n \in D} \left[\sum_{k \in K} A_k \xi^{-k} \right] u_n \xi^n - \sum_{r \in \Gamma} \left[\sum_{k \in K_r} A_k \xi^{-k} \right] u_r \xi^r = A(\xi) U(\xi) - \sum_{r \in \Gamma} A_r(\xi) u_r \xi^r$$

Hence

$$u_n = \frac{1}{(2\pi i)^s} \oint_{|\xi_s|=1} \cdots \oint_{|\xi_1|=1} \frac{U(\xi)}{\xi^{n+e}} d\xi_1 \cdots d\xi_s = \sum_{r \in \Gamma} \left[\frac{1}{(2\pi i)^s} \oint \cdots \oint \frac{A_r(\xi) u_r}{\xi^{n-r+e}} d\xi_1 \cdots d\xi_s \right] = \sum_{r \in \Gamma} \beta_n^r u_r.$$

Definition. We shall call the operator P_Γ the **boundary projection operator** if, to each vector-function v from U_Γ , it assigns the vector-function $u = P_\Gamma v$ from U_Γ by formulas (5), where n runs through the points of the set Γ .

Theorem 3. The operator P_Γ maps U_Γ onto the subspace of those vector-functions U'_Γ each of which is formed by the values of some solution of equation (1) at the points of the set Γ . Each vector-function from this subspace is mapped by P_Γ into itself.

Proof. The validity of the first assertion follows from Theorem 1, and of the second—from Theorem 2.

Corollary. The grid function u_Γ , $u_\Gamma \in U_\Gamma$, coincides with the values of some solution u of equation (1) at the points of the set Γ if and only if its values satisfy the system of equations

$$u_n = \sum_{r \in \Gamma} \beta_n^r u_r, \quad n \in \Gamma. \quad (6)$$

Theorem 4. There exist numbers C , $C > 0$, and q , $0 < q < 1$, independent of the domain D_0 , such that

$$\|\beta_n^r\| < C q^{|n-r|}, \quad r \in \Gamma, \quad n \in D, \quad (7)$$

where

$$|n - r| = |n_1 - r_1| + |n_2 - r_2| + \cdots + |n_s - r_s|.$$

Proof. We deform the circles $|\xi_j| = 1$, replacing them by the circles

$$|\xi_j| = q^{\text{sgn}(r_j - n_j)}, \quad j = 1, 2, \dots, s. \quad (8)$$

Here q , $q < 1$, can be chosen so close to unity that, on the manifold (8), as well as on the manifold (4), inequality (3) holds, and so that in the formula given

above for β_n^r one may pass from integration over the circles (4) to integration over the circles (8). Then

$$\|\beta_n^r\| = \frac{1}{(2\pi)^s} \left\| \dots \oint_{|\xi_j|=q^{\text{sgn}(r_j-n_j)}} \frac{A^{-1}(\xi)A_r(\xi)}{\xi^{n-r+e}} d\xi_1 \dots d\xi_s \right\| \leq \frac{\sup_{\xi} \|A^{-1}(\xi)A_r(\xi)\|}{(2\pi)^s} \prod_j \oint \left| \frac{1}{\xi^{n_j-r_j}} \right| \left\| \frac{d\xi}{\xi} \right\| = \sup_{\xi} \|A^{-1}(\xi)\|$$

Taking into account that

$$\|A_r(\xi)\| \leq M \sup_{k,\xi} \|A_k \xi^{-k}\|,$$

where M is the number of points forming the set K , one may take for C in (7)

$$C = M \sup_{\xi} \|A^{-1}(\xi)\| \sup_{k,\xi} \|A_k \xi^{-k}\|,$$

where ξ is a point on the manifold (8).

Thanks to Theorem 4, equations (6) can be approximated by a system of difference equations of fixed order, independent of the dimensions of the domain D (independent of the degree of refinement of the grid). This can be used for computing solutions of evolutionary implicit difference equations on graphs and polyhedra, for computing the spectrum of a family of difference operators on functions in multidimensional domains and on polyhedra, and also in certain other questions.

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Note: Figure translations are in progress. See original paper for figures.

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