

# On the Dual of a Subspace of a Countable Inductive Limit

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## Abstract

## Full Text

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MATHEMATICS

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# ON THE DUAL OF A SUBSPACE OF A COUNTABLE INDUCTIVE LIMIT

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Let  $E = \varinjlim E_n$ , where  $E_n$  are locally convex spaces, and let  $H$  be a vector subspace of  $E$ , endowed with the topology induced from  $E$ . There is a natural embedding

$$i : \varinjlim (E_n \cap H) \rightarrow H.$$

The question is posed: when is

$$i' : H' \rightarrow (\varinjlim (E_n \cap H))'$$

bijjective?

The corollary of Proposition 1 and Proposition 2 show the connection of this question with the surjectivity of the adjoint operator.

**Proposition 1.** Let  $E = \varinjlim E_n$ ,  $F = \varinjlim F_n$ , let  $E$  and  $F$  be separable, and let  $E_n$  and  $F_n$  be Fréchet spaces ( $n = 1, 2, 3, \dots$ ). Let  $T$  be a continuous linear operator mapping  $F$  into  $E$ , with sequentially closed image  $H$ . In order that  $T$  be a weak homomorphism of  $F$  onto  $H$ , it is necessary and sufficient that  $i'$  be bijective.

**Corollary.** Under the same assumptions, in order that  $T' : E' \rightarrow F'$  be surjective, it is necessary and sufficient that  $T$  be a monomorphism and that  $i'$  be bijective.

**Proposition 2.** Let  $F$  be a bornological space,  $E$  the strict inductive limit of an increasing sequence  $(E_n)$  of its closed subspaces, and let  $T$  be a continuous linear operator mapping  $F$  into  $E$ , with  $H = TF$ . If  $T' : E' \rightarrow F'$  is surjective, then  $i'$  is bijective.

An inverse spectrum  $(X_n)$ , with maps  $\alpha_n$ , is a sequence

$$\dots \xrightarrow{\alpha_n} X_n \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_{n-2}} \dots \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_1} X_1$$

of locally convex spaces  $X_n$  and continuous linear maps  $\alpha_n$ . A morphism of an inverse spectrum  $(X_n)$  with maps  $\alpha_n$  into an inverse spectrum  $(Y_n)$  with maps  $\beta_n$  is a set of continuous linear maps  $\gamma_n : X_n \rightarrow Y_n$  such that

$$\beta_n \gamma_{n+1} = \gamma_n \alpha_n \quad (n = 1, 2, 3, \dots).$$

Inverse spectra and their morphisms form a category. Dually, the category of direct spectra is defined. Consider the functor  $\text{Pro}$  from the category of inverse spectra to the category of linear spaces, assigning to each inverse spectrum its projective limit (without topology). The functor  $\text{Pro}$  is left exact; therefore one can speak of its derived functors  $\text{Pro}^1, \text{Pro}^2, \dots$ . These derived functors were computed in <sup>(2)</sup>.

$I$  is called an injective object of the category of locally convex spaces ( $TLC$ ) if a continuous linear mapping into  $I$  of a subspace of any locally convex space  $M$  can be extended to a continuous linear mapping of  $M$  into  $I$ .

Denote by  $\mathcal{L}(\cdot, A)$  the contravariant functor from the category  $TLC$  to the category of linear spaces, assigning to each object of  $TLC$  the linear space of its morphisms into a fixed object  $A$  of  $TLC$ . Then to the direct spectrum

$$\mathcal{E} : E_1 \rightarrow E_2 \rightarrow \dots \rightarrow E_n \rightarrow \dots$$

there corresponds

inverse spectrum

$$\mathcal{L}_A(\mathcal{E}) : \dots \rightarrow \mathcal{L}(E_n, A) \rightarrow \dots \rightarrow \mathcal{L}(E_2, A) \rightarrow \mathcal{L}(E_1, A).$$

With the aid of Theorem 11.2 from <sup>(2)</sup>, the following is proved.

**Lemma.** Let  $E = \varinjlim E_n$  and  $\forall n \exists p(n)$ : if  $q \geq p(n)$ , then the topologies induced in  $E_n$  from  $E_{p(n)}$  and  $E_q$  coincide. Then  $\text{Pro}^1 \mathcal{L}_I(\mathcal{E}) = 0$  for any injective object  $I$ .

Put  $G = E/H$ ,  $G_n = E_n/(E_n \cap H)$ ; let  $\tau_n$  be the quotient topology in  $G_n$ , and let  $\tau_{n,p}$  be the topology induced in  $G_n$  from  $G_{n+p}$ . Using the lemma and the fact that every locally convex space is isomorphic to a subspace of an injective object of the category  $TLC$  <sup>(2)</sup>, it is proved that

**Theorem 1.** In order that  $i : \varinjlim (E_n \cap H) \rightarrow H$  be an isomorphism, it is sufficient that

$$\forall n \exists p : q \geq p \Rightarrow \tau_{n,p} = \tau_{n,q}.$$

We shall denote by  $u_n^0$  and  $u_n^p$  the continuous linear functionals, respectively, on  $(G_n, \tau_n)$  and on  $(G_n, \tau_{n,p})$ .

**Theorem 2.** Let  $E_n$  be normed spaces and suppose the topologies induced in  $E_n$  from  $E_{n+1}$  and  $E_{n+2}$  coincide for all  $n$ . Denote by  $B_n$  the unit ball in  $E_n$  and by  $\widetilde{B}_n$  the projection of  $B_n$  in  $G_n$ . For bijectivity of  $i'$  it is necessary and sufficient that

$$\forall n \exists p : \forall u_n^p \forall q \geq p \forall \varepsilon > 0 \exists u_n^q : |u_n^p(x) - u_n^q(x)| \leq \varepsilon \forall x \in \widetilde{B}_n.$$

**Proof.** Consider the exact sequence of direct spectra  $0 \rightarrow (E_n \cap H) \rightarrow (E_n) \rightarrow (G_n) \rightarrow 0$  and pass to the algebraically exact sequence of inverse spectra  $0 \rightarrow (G'_n) \rightarrow (E'_n) \rightarrow ((E_n \cap H)') \rightarrow 0$ . Denoting the inverse spectra from left to right by  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ , we apply the functor  $\text{Pro}$ . We obtain the canonical exact sequence for a left-exact functor and its derived functors:

$$0 \rightarrow \text{Pro } \mathcal{L}_1 \rightarrow \text{Pro } \mathcal{L}_2 \rightarrow \text{Pro } \mathcal{L}_3 \rightarrow \text{Pro}^1 \mathcal{L}_1 \rightarrow \text{Pro}^1 \mathcal{L}_2 \rightarrow \dots$$

By the lemma,  $\text{Pro}^1 \mathcal{L}_2 = 0$ . Between  $\text{Pro } \mathcal{L}_1, \text{Pro } \mathcal{L}_2, \text{Pro } \mathcal{L}_3$  and, respectively,  $G', E', (\lim_{\rightarrow} (E_n \cap H))'$ , there are natural algebraic isomorphisms such that the following diagram is commutative:

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Pro } \mathcal{L}_1 & \rightarrow & \text{Pro } \mathcal{L}_2 & \rightarrow & \text{Pro } \mathcal{L}_3 \rightarrow \text{Pro}^1 \mathcal{L}_1 \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \rightarrow & G' & \rightarrow & E' & \rightarrow & (\lim_{\rightarrow} (E_n \cap H))' \rightarrow 0 \\ & & \parallel & & \parallel & & \uparrow i' \\ 0 & \rightarrow & G' & \rightarrow & E' & \rightarrow & H' \rightarrow 0 \end{array}$$

Since the lower row is exact, the bijectivity of  $i'$  is equivalent to the equality  $\text{Pro}^1 \mathcal{L}_1 = 0$ . To complete the proof of the theorem it is enough to introduce in each  $G'_n$  the strong topology and apply Corollary 11.4 from (2).

We shall denote by  $\bar{u}_k$  the continuous linear functionals on  $E$  whose restriction to  $E_k \cap H$  is zero, and by  $\bar{v}$  the continuous linear functionals on  $E$  whose restriction to  $H$  is zero.

**Corollary.** Let  $E$  be the strict inductive limit of a sequence  $(E_n)$  of normed subspaces. For bijectivity of  $i'$  it is necessary and sufficient that

$$\forall n \exists p : \forall \bar{u}_p \forall \varepsilon > 0 \exists \bar{v} : |\bar{u}_p(x) - \bar{v}(x)| \leq \varepsilon \forall x \in B_n.$$

A theorem analogous to Theorem 2 can also be proved in the case where the  $E_n$  have a countable fundamental system of bounded sets.

In the case where the  $E_n$  are arbitrary locally convex spaces, the following holds.

**Theorem 3.** Let  $E = \varinjlim E_n$ . For the bijectivity of  $i'$  it is sufficient that, for every  $n$ , there exist a neighborhood of zero  $U_n$  in  $E_n$  such that  $U_n \subset U_{n+1}$ ,

$$\forall n \exists p : \forall u_n^p \forall q \geq p \forall \varepsilon > 0 \exists u_n^q : |u_n^p(x) - u_n^q(x)| \leq \varepsilon \quad \forall x \in \widetilde{U}_n,$$

where  $\widetilde{U}_n$  is the image of  $U_n$  in  $G_n$ .

**Corollary 1.** Let  $E$  be a strict inductive limit of a sequence  $(E_n)$  of its subspaces. For the bijectivity of  $i'$  it is sufficient that there exist a neighborhood of zero  $U$  in  $E$  such that

$$\forall n \exists p : \forall \bar{u}_p \forall \varepsilon > 0 \exists \bar{v} : |\bar{u}_p(x) - \bar{v}(x)| \leq \varepsilon \quad \forall x \in E_n \cap U.$$

**Corollary 2.** Let  $E = \varinjlim E_n$ , where the mappings of  $E_n$  into  $E_{n+1}$  are weakly completely continuous, and the  $E_n$  are separable ( $n = 1, 2, \dots$ ). If  $E_n \cap H$  is closed in  $E_n$  for all  $n$ , then  $i'$  is bijective.

In the case where the  $E_n$  are reflexive Banach spaces, the result of Corollary 2 was obtained by Foiaş and Marinescu <sup>(3,4)</sup>, and for closed  $H$ , by Komatsu <sup>(1)</sup>.

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*Note: Figure translations are in progress. See original paper for figures.*

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