

ON THE LOCATION OF THE POINTS OF MAXIMAL DEVIATION IN CHEBYSHEV APPROXIMATION OF CONTINUOUS FUNCTIONS

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Abstract

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MATHEMATICS

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ON THE LOCATION OF THE POINTS OF MAXIMAL DEVIATION IN CHEBYSHEV APPROXIMATION OF CONTINUOUS FUNC- TIONS

(Presented by Academician S. N. Bernstein on 6 XI 1968)

Let $f(t) \in C_{2\pi}$; let $T_n(t; f)$ be the trigonometric polynomial of order $\leq n$ least deviating from $f(t)$ in the metric of the space $C_{2\pi}$;

$$E_n^*(f) = \max |f(t) - T_n(t; f)| \quad (0 \leq t \leq 2\pi);$$

let $M_n^*(f)$ be the set of points $t \in [0, 2\pi)$ where

$$E_n^*(f) = |f(t) - T_n(t; f)|.$$

Put

$$M^*(f) = \bigcup_{n=0}^{\infty} M_n^*(f).$$

For $f(x) \in C[-1, 1]$, $E_n(f)$ and $M(f)$ are defined analogously.

In paper ⁽¹⁾ M. I. Kadets formulated a theorem which is a generalization of Theorem II ⁽²⁾, p. 88, of S. N. Bernstein, and from which it follows that the set $M(f)$ of any function $f(x) \in C[-1, 1]$ is everywhere dense on $[-1, 1]$. The aim of the present note is to prove that the set $M^*(f)$ of any function $f(t) \in C_{2\pi}$ is everywhere dense on the interval $[0, 2\pi]$.

As noted in ⁽¹⁾, the series

$$\sum_{n=0}^{\infty} \frac{E_n^*(f) - E_{n+1}^*(f)}{E_n^*(f) + E_{n+1}^*(f)}$$

diverges, whence it follows that

$$\overline{\lim}_{n \rightarrow \infty} \left(\frac{E_n^*(f) - E_{n+1}^*(f)}{E_n^*(f) + E_{n+1}^*(f)} \right)^{1/n} = 1.$$

Lemma 1. If $f(t) \in C_{2\pi}$ and $E_p^*(f) = E_{r-1}^*(f) > E_r^*(f)$ ($p < r - 1$), then the polynomial $T_r^*(t) = T_p(t; f) - T_r(t; f)$ has $2r$ simple zeros $z_1 < z_2 < \dots < z_{2r} \in (0, 2\pi)$; the points $\{u_i\}_{i=1}^{2r}$ of maximal deviation from zero of the difference $f(t) - T_p(t; f)$ alternate with the zeros $\{z_i\}$ of the polynomial $T_r^*(t)$.

Since $E_p^*(f) = E_{r-1}^*(f)$, we have $T_p(t; f) = T_{r-1}(t; f)$.

For the proof of Lemma 1 it suffices to note that

$$T_p(t; f) - T_r(t; f) = [T_{r-1}(t; f) - f(t)] - [T_r(t; f) - f(t)]$$

and that $E_{r-1}^*(f) > E_r^*(f)$.

Denote by $W_n(h)$ the class of trigonometric polynomials $T(t)$ of order n having the following properties:

- 1) the polynomial $T(t)$ has, in $(0, 2\pi)$, $2n$ simple zeros

$$z_1 < z_2 < \dots < z_{2n};$$

- 2)

$$\min_{0 \leq i \leq 2n} \|T\|_{[z_i, z_{i+1}]} \geq h \|T\|_{[0, 2\pi]} \quad (z_0 = 0, z_{2n+1} = 2\pi; 0 < h \leq 1).$$

Lemma 2. Let $x_1, x_2, \dots, x_{2n} \in (0, 2\pi)$ be the zeros of a polynomial $T(t) \in W_n(h)$; let $t_1, t_2, \dots, t_{2n} \in [0, 2\pi]$ be the zeros of its derivative $T'(t)$; let $y_1, y_2, \dots, y_{2n} \in (0, 2\pi)$ be the zeros of a polynomial $Q(t) \in W_n(h)$; and let $q_1, q_2, \dots, q_{2n} \in [0, 2\pi]$ be the zeros of its derivative $Q'(t)$. If $T(t_i) = Q(q_i)$ for $i = 0, 1, \dots, j-1, j+1, \dots, 2n$ and $T(t_j) > Q(q_j) > 0$ ($T(t_j) < Q(q_j) < 0$), then

$$0 \leq t_1 < q_1 < t_2 < q_2 < \dots < t_{j-1} < q_{j-1} < q_{j+1} < t_{j+1} < q_{j+2} <$$

$$< t_{j+2} < \dots < q_{2n} < t_{2n} < 2\pi; \quad (1)$$

$$0 < x_1 < y_1 < x_2 < y_2 < \dots < x_j < y_j < y_{j+1} < x_{j+1} < \dots$$

$$\dots < y_{2n} < x_{2n} < 2\pi; \quad (2)$$

$$y_1 - x_1 < y_2 - x_2 < \dots < y_j - x_j;$$

$$x_{j+1} - y_{j+1} > x_{j+2} - y_{j+2} > \dots > x_{2n} - y_{2n}. \quad (3)$$

As in the proof of Lemma 1 in (3), here one can also indicate the process of passing from the polynomial $T(t)$ to the polynomial $Q(t)$, from which (1) and (2) will follow.

Let us prove assertion (3). Suppose that, for some fixed i ($1 < i \leq j$), $y_i - x_i \leq y_{i-1} - x_{i-1}$. By the condition of Lemma 2, on the interval $[0, 2\pi)$ the trigonometric polynomial

$$T(t) - Q(t + (y_{i-1} - x_{i-1})) = F(t)$$

of order n must have at least i zeros in the half-interval $[0, x_i)$, and at least $2n - i$ zeros in the interval $(x_i, 2\pi)$; moreover, x_i is a zero of $F(t)$. Since $F(t)$ can have $\leq 2n$ zeros in the interval $[0, 2\pi)$, our supposition is false.

Denote by $\tau(t; i)$ ($i = 0, 1, \dots, 2n$) the trigonometric polynomial $\in W_n(h)$ such that

$$\tau(\tau_l^i; i) = \begin{cases} (-1)^{2n-l} & \text{for } l = 0, 1, \dots, i-1, \\ (-1)^{2n-l} & \text{for } l = i+1, \dots, 2n, \end{cases}$$

and $\tau'(\tau_j^i; i) = 0$ for $j = 0, 1, \dots, 2n$, where $\tau_j^i \in [0, 2\pi)$. The existence and uniqueness of such polynomials are proved in (4). The polynomial $\tau(t; n)$ has the form (see, for example, (5, 6))

$$\tau(t; n) = h \cos 2n \arccos \left(\frac{\sin(t/2)}{\sin(\omega_n/2)} \right),$$

where ω_n is chosen so that $\tau(\pi; n) = 1$, i.e., from the condition

$$\operatorname{tg}^{2n} \omega_n / 4 + \operatorname{ctg}^{2n} \omega_n / 4 = 2/h. \quad (4)$$

For the polynomials $\{\tau(t; i)\}_{i=0}^{2n}$ the equality

$$\tau(t; i) = (-1)^{n-i} \tau(t + \alpha_i; n),$$

holds, where α_i is a constant depending on i .

Put

$$d_n(T; h) = \max |z_i^T - z_{i+1}^T|, \quad 0 \leq i \leq 2n; \quad D_n(h) = \sup_{T(t) \in W_n(h)} d_n(T; h),$$

where $z_1^T, z_2^T, \dots, z_{2n}^T \in (0, 2\pi)$ are the zeros of the polynomial $T(t) \in W_n(h)$, $z_0^T = 0, z_{2n+1}^T = 2\pi$.

Theorem 1. The upper bound $D_n(h)$ is attained on the polynomials $\tau(t; i)$ and is equal to

$$D_n(h) = 2 \left(\pi - 2 \arcsin \left(\frac{\sin \omega_n}{2 \cos(4\pi - 1)\pi/8n^2} \right) \right) = O(1/\sqrt[3]{h}).$$

We shall show that $D_n(h)$ cannot be attained on a polynomial $T(t) \in W_n(h)$ for which $\|T\|_{C_{2\pi}} < 1$. Suppose, to the contrary, that

$$D_n(h) = |z_i^T - z_{i+1}^T|,$$

where z_i^T and z_{i+1}^T are neighboring zeros of the polynomial $T(t)$. Let

$$t_0 < t_1 < \dots < t_{2n} \in [0, 2\pi)$$

be the zeros of the derivative $T'(t)$ of the polynomial $T(t)$, and suppose, for definiteness, that $T(t_i) > 0$, where $z_i^T < t_i < z_{i+1}^T$.

Consider a polynomial $S(t) \in W_n(h)$ such that $S(s_j) = T(t_j)$ for

$$j = 0, 1, \dots, i-1, i+1, \dots, 2n;$$

$S(s_i) = 1$, where the s_j are the zeros of the derivative $S'(t)$ of the polynomial $S(t)$, lying in the interval $[0, 2\pi)$. Let z_i^S and $z_{i+1}^S \in (0, 2\pi)$ be neighboring zeros of the polynomial $S(t)$. Then from Lemma 2 we have

$$z_i^S < z_i^T \quad \text{and} \quad z_{i+1}^T > z_{i+1}^S,$$

or

$$D_n(h) = |z_i^T - z_{i+1}^T| < |z_i^S - z_{i+1}^S|.$$

Show that $D_n(h)$ also cannot be attained on a polynomial $Q(t) \neq \tau(t; i)$ from $W_n(h)$. Suppose, to the contrary, that $D_n(h)$ is attained on a polynomial $Q(t) \neq \tau(t; i)$ for the zeros z_i^q and z_{i+1}^q , i.e.,

$$D_n(h) = |z_i^q - z_{i+1}^q|.$$

If $Q(t) \neq \tau(t; i)$, then at least one of the inequalities

$$Q(q_l) \geq \tau(\tau_l^i; i) \quad (l = 0, 1, \dots, 2n),$$

holds, where $q_l \in [0, 2\pi)$ and $\tau_l^i \in [0, 2\pi)$ are, respectively, points of extremum of the polynomials $Q(t)$ and $\tau(t; i)$. For definiteness, let

$$Q(q_j) > \tau(\tau_j^i; i) > 0 \quad (j \neq i).$$

Then from Lemma 2, by virtue of inequalities (3), we have

$$z_i^T < z_i^q < z_{i+1}^q < z_{i+1}^T$$

or

$$|z_i^q - z_{i+1}^q| < |z_i^r - z_{i+1}^r|,$$

where z_i^r and z_{i+1}^r are the i -th and $(i+1)$ -st zeros of the polynomial $\tau(t; i)$ on the interval $[0, 2\pi]$. The quantity $D_n(h)$ is determined by direct calculations. Theorem 1 is proved.

Let the sequence of numbers $\{h_n\}_{n=1}^\infty$ ($0 < h_n < 1$, $n = 1, 2, \dots$) be such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{h_n} = 1; \quad (5)$$

$\{W_n(h_n)\}_{n=1}^\infty$ is the sequence of classes of polynomials corresponding to $\{h_n\}_{n=1}^\infty$; $\{T_n(t)\}_{n=1}^\infty = \mathcal{T}$ is a sequence of polynomials such that $T_n(t) \in W_n(h_n)$ for any $n = 1, 2, \dots$. Denote by Z_n the set of zeros of the polynomial $T_n(t) \in W_n(h_n)$ situated in the interval $[0, 2\pi)$, and put $Z = \bigcup_{n=1}^\infty Z_n$. We shall call Z the set of zeros of $\mathcal{T}$.

Theorem 2. *The set Z of zeros of the sequence $\mathcal{T}$ is everywhere dense on the interval $[0, 2\pi]$.*

We have $|z_{i,n}^T - z_{i+1,n}^T| \leq D_n(h_n)$. By (5), it follows from (4) that $\lim_{n \rightarrow \infty} \omega_n = \pi$ as $n \rightarrow \infty$. Hence, and from Theorem 1, it follows that $\lim_{n \rightarrow \infty} D_n(h_n) = 0$ as $n \rightarrow \infty$. Theorem 2 is proved.

Theorem 3. *The set $M^*(f)$ of functions $f(t) \in C_{2\pi}$ is everywhere dense on the interval $[0, 2\pi]$.*

The polynomial

$$T_{n_{k+1}}^*(t) = T_{n_k}(t; f) - T_{n_{k+1}}(t; f) \in W_{n_{k+1}}(h_{n_{k+1}}),$$

where

$$h_{n_{k+1}} = \{E_{n_k}^*(f) - E_{n_{k+1}}^*(f)\} / \{E_{n_k}^*(f) + E_{n_{k+1}}^*(f)\} \quad (k = 0, 1, 2, \dots).$$

Since $f(t) \in C_{2\pi}$, we have

$$\lim_{k \rightarrow \infty} \sqrt[n_k]{h_{n_k}} = 1$$

as $k \rightarrow \infty$. Therefore, by Theorem 2, the set Z of zeros of the sequence $\{T_{n_{k+1}}^*(t)\}_{k=0}^\infty$ is everywhere dense on $[0, 2\pi]$. By virtue of Lemma 1, the zeros of $T_{n_{k+1}}^*(t)$ alternate with the points of maximum deviation from zero of the difference $f(t) - T_{n_k}(t, f)$ for every $k = 0, 1, 2, \dots$. Theorem 3 is proved.

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Note: Figure translations are in progress. See original paper for figures.

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