

NONLINEAR EQUATIONS WITH MONOTONE OPERATORS

A. M. Vainberg

1969

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196901.86217>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.3

MATHEMATICS

A. M. VAINBERG

NONLINEAR EQUATIONS WITH MONOTONE OPERATORS

(Presented by Academician G. I. Petrov on 19 II 1969)

In the present paper, existence and uniqueness theorems are established for solutions of nonlinear integral equations of Hammerstein type in the Lebesgue spaces $L_{p,n}(G)$.

In works known to the author, in the study of the Hammerstein equation $u + Ahu = 0$, where A is a linear integral operator and h is a Nemytskii operator, it was required that the Hammerstein operator Ah be completely continuous; and, when this requirement was abandoned, it was assumed that A is a self-adjoint operator, $p \geq 2$, $\text{mes } G < \infty$. In the present paper it has been possible to dispense both with these requirements and with some others. In the first part, a general existence and uniqueness theorem is established for systems of Hammerstein equations, and in the last part an analogous theorem for an abstract equation of Hammerstein type.

1. Let G be a measurable set of positive (finite or infinite) measure in s -dimensional Euclidean space; let $g_i(u_1, u_2, \dots, u_n, x)$ be real functions, continuous jointly in $(u_1, u_2, \dots, u_n)$ for almost every $x \in G$ and measurable in G with respect to x for fixed $(u_1, u_2, \dots, u_n)$, $-\infty < u_i < +\infty$ ($i = 1, 2, \dots, n$). Consider the system of nonlinear integral equations

$$u_i(x) + \sum_{j=1}^n \int_G K_{ij}(x, y) g_j(u_1(y), u_2(y), u_3(y), \dots, u_n(y), y) dy \quad (1)$$

under the assumption that each of the kernels $K_{ij}(x, y)$ generates a bounded integral operator B_{ij} :

$$B_{ij}v = \int_G K_{ij}(x, y)v(y) dy$$

from the space $L^q(G)$ into $L^p(G)$, $p > 1$, $p^{-1} + q^{-1} = 1$.

Theorem 1. If the matrix $K = (K_{ij}(x, y))$ generates a positive operator from $L_{q,n}$ into $L_{p,n}$, i.e.

$$f(v) = \sum_{i,j=1}^n \int_G \int_G K_{ij}(x, y) v_i(x) v_j(y) dx dy \geq 0, \quad v_i \in L^q,$$

and the functions g_i satisfy the conditions

$$|g_i(u_1, u_2, \dots, u_n, x)| \leq a_i(x) + b \sum_{k=1}^n |u_k|^{p-1}, \quad a_i(x) \in L^q, \quad b > 0, \quad (2)$$

$$\sum_{i=1}^n [g_i(u_1, u_2, \dots, u_n, x) - g_i(v_1, v_2, \dots, v_n, x)](u_i - v_i) \geq 0 \quad (3)$$

for almost every $x \in G$ and arbitrary u_i and v_i ,

$$u_i g_i(u_1, u_2, \dots, u_n, x) \geq \alpha |u_i|^p - \sum_{k=1}^n |\beta_{ik}(x)| |u_k|^p - |c_i(x)|, \quad (4)$$

where $a > 0$, $1 \leq \gamma < p$, $\beta_{ik}(x) \in L^t$, $t = p/(p - \gamma)$, $c_i(x) \in L(G)$, then system (1) has a solution belonging to $L_{p,n}(G)$. This solution is unique if K generates a strictly positive operator or if inequality (3) is strict.*

The proof uses the properties of the Nemytskii operator h , generated by the functions g_i (1), the weak lower semicontinuity of the functional $f(v)$ in the space $L_{q,n}$, and the Browder-Minty lemma (2, 3).

Remark 1. The assertion of Theorem 1 remains valid if inequality (4) is replaced by the requirement: there exists a positive number such that, as soon as $\|u\| > \lambda$,

$$(hu, u) \geq 0,$$

where h is the Nemytskii operator generated by the functions $g_i(u_1, u_2, \dots, u_n, x)$, and (v, u) is the value of the linear functional $v \in L_{q,n}$ on the vector $u \in L_{p,n}$.

2. Let the kernel $K(x, y)$ generate a linear bounded positive operator B :

$$Bv = \int_G K(x, y) v(y) dy$$

from the space L^q into L^p , $p > 1$, $p^{-1} + q^{-1} = 1$, and let $g(u, x)$ be a function continuous in u for almost every $x \in G$ and measurable in G with respect to x for every $u \in (-\infty, +\infty)$. From Theorem 1 it follows

Theorem 2. If the function $g(u, x)$, nondecreasing in u for almost every $x \in G$, satisfies the condition

$$|g(u, x)| \leq a(x) + b|u|^{p-1}, \quad a(x) \in L^q,$$

and there exists $\lambda > 0$ such that

$$\int_G g(u(x), x)u(x) dx \geq 0 \quad \text{for } \|u\| > \lambda,$$

then the nonlinear integral equation

$$u(x) + \int_G K(x, y)g(u(y), y) dy = 0$$

has a solution belonging to L^p . This solution is unique if $g(u, x)$ strictly increases with respect to u , or when the kernel $K(x, y)$ generates a strictly positive integral operator.

3. We shall say that a real Banach space E has property (π) if the following conditions are fulfilled: 1) E is reflexive; 2) there exists a monotone hemicontinuous operator A from E^* into E such that $A0 = 0$, Au is continuous at zero of the space, and $(Au, u) \geq c\|u\|^\alpha$, where $\alpha > 1$.

Examples of spaces with property (π) are reflexive Banach spaces E in which the norm in E^* is Fréchet differentiable. In such spaces the operator A can be given by the formula

$$Ax = \|x\|^\alpha \operatorname{grad} \|x\|, \quad \alpha > 0, \quad x \neq 0, \quad x \in E^*; \quad A0 = 0,$$

since

$$(Ax, x) = \|x\|^{\alpha+1}; \quad (Ax - Ay, x - y) \geq (\|x\| - \|y\|)(\|x\|^\alpha - \|y\|^\alpha) \geq 0.$$

Theorem 3. Let the Banach space E have property (π) , let F be a monotone hemicontinuous and bounded operator from E into E^* such that

$$(Fu, u) \geq 0, \quad \text{if } \|u\| > \lambda > 0,$$

and let B be a linear bounded and positive operator from E^* into E .

* That is, equality almost everywhere in x takes place only when $u_i = v_i$ for all i .

Then the equation

$$u + BFu = 0$$

has a solution belonging to E . This solution is unique when B is a strictly positive operator or when F is a strictly monotone operator.

Let us note that from this theorem follows Theorem 4.7, established in paper ⁽⁴⁾ for a Hilbert space.

State Scientific-Research
and Design Institute of the Nitrogen Industry

Received
13 II 1969

REFERENCES

- ¹ M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, 1956.
- ² F. E. Browder, Bull. Am. Math. Soc., **69**, 862 (1963).
- ³ G. J. Minty, Proc. Nat. Acad. Sci. U. S. A., **50**, 1038 (1963).
- ⁴ C. L. Dolph, G. J. Minty, On Nonlinear Integral Equations of the Hammerstein Type, Wisconsin, Madison, 1964.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.