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Abstract

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MATHEMATICS

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ON THE PROBLEM OF OPTIMIZING ALGORITHMS FOR APPROXIMATE INTEGRATION OF FUNCTIONS OF MANY VARIABLES

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1°. In this note we shall consider questions of optimizing the cubature process on certain classes of functions. Let x denote a variable point of the n -dimensional Euclidean space E_n , and ξ a variable point of the n -dimensional Euclidean space E^n . By D , as usual, we shall denote the set of functions from C^∞ with compact supports, and by D' the corresponding set of distributions. The objects of our study will be several functional spaces: the space $H_2^{(\mu)}(E_n)$, consisting of distributions $u \in D'$ whose Fourier transforms $\hat{u}(\xi)$ are square-integrable with weight $\mu^2(\xi)$; the space $H_2^{(\mu)}(\Omega)$, whose elements are restrictions of functions belonging to the space $H_2^{(\mu)}(E_n)$ to a bounded domain $\Omega \subset E_n$; and also the space $\widetilde{H}_2^{(\mu)}(E_n)$, consisting of distributions $u \in D'$, periodic with a fundamental period matrix T (T -periodic), whose Fourier coefficients $\hat{u}(\beta)$ are square-summable with the discrete weight $\mu^2(\beta T^{-1})$ (β is an n -dimensional integer vector). We introduce the topology in these spaces as follows:

$$\|u\|_{H_2^{(\mu)}(E_n)} = \left(\int_{E_n} |\hat{u}(\xi)|^2 \mu^2(\xi) d\xi \right)^{1/2}, \quad (1)$$

$$\|u\|_{H_2^{(\mu)}(\Omega)} = \inf \|u^n\|_{H_2^{(\mu)}(E_n)} \quad (2)$$

(the lower bound is taken over all such elements $u^n \in H_2^{(\mu)}(E_n)$ whose restriction to Ω coincides with u , i.e., over all extensions of u to the space E_n),

$$\|u\|_{\widetilde{H}_2^{(\mu)}(E_n)} = \left(|T| \sum_{\beta} |\hat{u}(\beta)|^2 \mu^2(\beta T^{-1}) \right)^{1/2} \quad (3)$$

($|T|$ denotes the determinant of the matrix T).

It is clear that, in order to develop a meaningful theory of cubature formulas in these spaces, certain conditions must be imposed on the weight functions μ . We shall consider two classes of weight functions. The class B_n will be defined as the set of functions $\mu(\xi)$, continuous in E^n , generating spaces embedded in the space C . The class $B_n^{(m)}$ will be defined as the set of those functions μ from B_n which satisfy the conditions:

1. $\mu(\xi)$ is homogeneous of degree m , ($m > n/2$). (4)

2. $|D^\alpha F[\mu^{-2}(\xi)]| \leq K|x|^{2m-n-|\alpha|}$. (5)

($\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \sum \alpha_i$, $|x| = (\sum x_i^2)^{1/2}$.) Associating with the cubature formula

$$\int_{\Omega} \varphi(x) dx \simeq \sum_{k=1}^N c_k \varphi(x^{(k)}) \quad (6)$$

error functional

$$(l, \varphi) \equiv \int_{\Omega} \varphi(x) dx - \sum_{k=1}^N c_k \varphi(x^{(k)}), \quad (7)$$

we can, by virtue of the assumptions made, characterize its quality by the norm of this functional

$$\|l\|_{\Phi} = \sup_{\varphi} |(l, \varphi)| / \|\varphi\|_{\Phi}, \quad (8)$$

where Φ is one of the three spaces indicated above.

Finding the minimum of the norm (8) with respect to $c_k, x^{(k)}$ is a typical min-max problem; the c_k and $x^{(k)}$ realizing the minimax give the optimal formula. This problem is so complicated that even for functions of one variable it has been solved only in a few special cases. In view of this we shall confine ourselves to considering its simplified variant; namely, we shall optimize the norm (8) only with respect to the coefficients, assuming the system of nodes to be fixed. As shown in ⁽¹⁾, in this case the problem degenerates into a linear problem, since for determining the optimal coefficients we obtain a system of linear algebraic equations. Thus, from the theoretical point of view, the question could be regarded as exhausted. However, in practice, when computing integrals of high multiplicity, we shall have to deal with a system of very high order, and its direct solution will be quite difficult. It therefore seems expedient to continue the investigation of the problem formulated above.

2°. Let us first consider the simplest periodic case. We shall now show that, for a certain method of choosing the nodes, optimal periodic formulas can be obtained from considerations of an entirely different nature than those discussed above. We shall consider cubature formulas with a system of nodes of the form

$$x^{(\beta)} = hH\beta, \quad (9)$$

where β is an n -dimensional integer vector, H is a matrix with determinant equal to one, and h is some small parameter. The error functional in this case will have the form

$$\tilde{l}(x) = 1 - \sum_{\beta} c_{\beta} \delta\left(\frac{x}{h} - H\beta\right). \quad (10)$$

Theorem 1. *In the optimal periodic formula all coefficients are equal to one another, i.e., the optimal formula is a formula with equal coefficients.*

The proof of this theorem for the space $\tilde{H}_2^{(\mu)}(E_n)$ is in no way different from the proof of S. L. Sobolev, which he carried out for his space $\tilde{L}_2^{(m)}(E_n)$ (see (2)), and follows easily from the fact that in any Hilbert space the sphere is strictly convex.

Let us now calculate the error yielded by the optimal formulas. Suppose that the torus $\tilde{\Omega}$, over which the integral is computed, is determined by a matrix that is a multiple of the lattice matrix.

Theorem 2. *The norm of the error functional $\tilde{l}_0$ of the optimal cubature formula in the space $\tilde{H}_2^{(\mu)}(E_n)$ ($\mu \in B_n^{(m)}$) has the form*

$$\|\tilde{l}_0\|_{\tilde{H}_2^{(\mu)*}(E_n)} = (h/2\pi)^m \sqrt{B_n^{(\mu)}(H)} \sqrt{V(\tilde{\Omega})}, \quad (11)$$

where

$$B_n^{(\mu)}(H) = \sum_{\beta \neq 0} \frac{1}{\mu^2(\beta H^{-1})}, \quad (12)$$

$V(\tilde{\Omega})$ is the volume of the domain $\tilde{\Omega}$.

For the proof of this theorem, by the method indicated in note (3), one constructs the extremal function of the cubature formula, i.e., the function on which the error functional of the formula attains its greatest value on the sphere of unit radius in the space $H_2^{(\mu)}(E_n)$. It will have the form

$$\tilde{u}_0(x) = \sum_{\beta \neq 0} \frac{1}{\mu^2(\beta h^{-1}H^{-1})} \exp(2\pi i \beta h^{-1}H^{-1}x). \quad (13)$$

This already makes it possible, in an elementary way, using the properties of the weight class $B_n^{(m)}$, to obtain formula (11).

We have restricted ourselves to lattice distributions of nodes. Of course, we can further vary the choice of lattices and, among all lattice structures, choose the best one, although, naturally, it will not follow from anywhere that this latter will be the best structure of nodes in general. Formula (11) shows that the best or optimal lattice will be the one for which the constant $B_n^{(\mu)}(H)$ is minimal. We note that finding such lattices is a nontrivial problem of the geometric theory of numbers.

3°. In the general case of the spaces $H_2^{(\mu)}(\Omega)$ we shall find asymptotically optimal formulas, i.e., formulas that have the same leading term in the norm of the error functional as the optimal ones.

We shall use the formulas constructed by S. L. Sobolev (see (2)) with a regular boundary layer. Recall that, according to S. L. Sobolev, a formula is called a formula with a regular boundary layer of order p if its error functional admits the representation

$$l(x) = \sum_{\gamma} l_{hH}^{(\gamma)}(x), \quad (14)$$

in which all component functionals have the form

$$l_{hH}^{(\gamma)}(x) = \mathcal{E}_{\Omega_{hH}^{(\gamma)}}(x) - \sum_{|\gamma'| < L} C_{\gamma'}^{(\gamma)} \delta(x - hH(\gamma + \gamma')) \quad (15)$$

($\Omega_{hH}^{(\gamma)}$ is the intersection of the domain obtained from the parallelepiped Ω_0 , corresponding to the matrix H ($|H| = 1$), by shifting by the vector γ and dilating h times, with the domain Ω) and satisfy the conditions

$$\text{supp}(l_{hH}^{(\gamma)}(x)) \subset \mathcal{E}(|x - hH\gamma| < Lh); \quad (16)$$

$$\|l_{hH}^{(\gamma)}(x)\|_{C^*} \leq Kh^n; \quad (17)$$

$$(l_{hH}^{(\gamma)}, x^\alpha) = 0, \quad \text{if } |\alpha| < p \quad (18)$$

($\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \sum_{i=1}^n \alpha_i$). We note that in any such formula all coefficients corresponding to nodes more than $2Lh$ away from the boundary coincide with one another and are equal to h^n . Therefore, in the sense of construction, these formulas are considerably simpler than the optimal ones, since their coefficients need be computed only in the boundary layer.

Theorem 3. In the space $H_2^{(\mu)}(\Omega)$ ($\mu \in B_n^{(m)}$), the norm of the error functional of any cubature formula with a regular boundary layer has the form

$$\|l\|_{H_2^{(\mu)*}(\Omega)} = (h/2\pi)^m (B_n^{(\mu)}(H))^{1/2} (V(\Omega))^{1/2} + O(h^{m+1}). \quad (19)$$

Let us outline the proof of this theorem. According to ⁽³⁾, the extremal function corresponding to the functional $l(x)$ in the case of the space $H_2^{(\mu)}(\Omega)$ will be the function

$$u_0(x) = l(x) * v(x) \quad (v(x) = F[\mu^{-2}(\xi)]). \quad (20)$$

Representing $l_0(x)$ in the form of the difference

$$l(x) = \tilde{l}_0(x) - l'(x), \quad (21)$$

where $\tilde{l}_0(x)$ is the optimal periodic functional, and $l'(x)$ is the error functional of a cubature formula with a regular boundary layer for the domain $\Omega' = E_n \setminus \Omega$, we obtain for the extremal function an analogous representation

$$u_0(x) = \tilde{u}_0(x) - u'_0(x), \quad (22)$$

where $\tilde{u}_0(x)$ is determined by formula (13), and $u'_0(x) = l'(x) * \nu(x)$. Consequently, the square of the norm of our error functional can be written in the form

$$\|l(x)\|_{H_2^{(\mu)*}(\Omega)}^2 = (l(x), \tilde{u}_0(x)) - (l(x) * \nu(x) * l'(-x))|_{x=0}. \quad (23)$$

By means of the function $\tilde{u}_0(x)$ we single out the principal term; all the rest gives a remainder of order $O(h^{2m+1})$. The latter fact is a consequence of relations (5), (14)–(18).

Theorem 4. *Formulas with a regular boundary layer of order m in the classes $H_2^{(\mu)}(\Omega)$ ($\mu \in B_n^{(\mu)}$) are asymptotically optimal.*

At present several methods are known for proving theorems of this type (see ^(2–4)). The simplest of them is given in work ⁽⁴⁾. We shall mainly follow this last variant, modifying it only slightly. In view of relation (19), the theorem will be proved if we establish that for any functional in the class $H_2^{(\mu)}(\Omega)$ (including the optimal one) the following lower estimate is valid:

$$\|l(x)\|_{H_2^{(\mu)*}(\Omega)} \geq h^m (B_n^{(\mu)}(H))^{1/2} (V(\Omega))^{1/2} + O(h^{m+1}). \quad (24)$$

Consider the function

$$v(x) = (\tilde{u}_0(0) - \tilde{u}_0(x))\eta(x), \quad (25)$$

where $\eta(x)$ is a finite function equal to unity in the domain Ω^* , obtained by uniting all domains $\Omega_{\gamma h}^{(v)}$ having an intersection with the domain Ω . It is not difficult to see that

$$\|u\|_{H_2^{(\mu)}(\Omega)} = \inf_{u^n} \|u^n\|_{H_2^{(\mu)}(E_n)} \leq \|v\|_{H_2^{(\mu)}(E_n)} \leq h^m (B_n^{(\mu)}(H))^{1/2} (V(\Omega))^{1/2} + O(h^{m+1}). \quad (26)$$

On the other hand, by the very choice of the function, its values at the nodes are equal to zero. Hence it follows easily that

$$|(l, v)| = h^{2m} B_n^{(\mu)}(H) V(\Omega) + O(h^{2m+1}). \quad (27)$$

Estimate (24) now follows trivially from relations (26), (27). Thus, S. L. Sobolev's formulas turn out to be universally asymptotically optimal in the class of spaces $H_2^{(\mu)}(\Omega)$ in the sense of I. Babushka (see (6)).

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