

THE SCHAUDER PRINCIPLE AND THE STABILITY OF PERIODIC SOLUTIONS

Yu. S. Kolesov

1969

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196901.83636>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.948:513.88+517.919.2

MATHEMATICS

Yu. S. KOLESOV

THE SCHAUDER PRINCIPLE AND THE STABILITY OF PERIODIC SOLUTIONS

(Presented by Academician I. G. Petrovskii on 12 III 1969)

This note studies the connection between the Schauder principle and the stability of periodic solutions of nonlinear differential equations. The investigation is based on the following two general theorems.

Let F be a completely continuous operator acting in a real Banach space E . A fixed point x_0 of this operator is called stable if, for every $\varepsilon > 0$, one can specify a $\delta > 0$ such that from the inequality $\|x - x_0\| \leq \delta$ there follow the inequalities $\|F^n x - x_0\| \leq \varepsilon$ ($n = 1, 2, \dots$). The point x_0 is called asymptotically stable if, in addition, $\|F^n x - x_0\| \rightarrow 0$ ($n \rightarrow \infty$).

Assume first that $F\theta = \theta$ (θ is the zero of the space E) and that the operator F is representable in the form

$$Fx = Ax + Bx,$$

where A is a linear operator, while the operator B satisfies the condition

$$\|Bx_1 - Bx_2\| \leq q(r)\|x_1 - x_2\| \quad (\|x_1\|, \|x_2\| \leq r),$$

where $q(r) \rightarrow 0$ ($r \rightarrow 0$). Further, suppose that 1 is a simple eigenvalue of the operator A , and that all other points of the spectrum of this operator lie inside the unit circle.

Theorem 1. *The point θ is asymptotically stable if and only if it is an isolated fixed point of the vector field*

$$\Phi x = x - Fx \tag{1}$$

and if its index is equal to 1.

The proof of the theorem is carried out by methods of the theory of cones ⁽¹⁾. Along the way it is shown that the index of the fixed point θ of the vector field (1) can be equal only to 0, 1, or -1 .

Assume now that the Banach space E is semi-ordered by means of some normal cone K . By $\langle x_1, x_2 \rangle$ we shall denote the set of elements x satisfying the inequalities

$$x_1 \leq x \leq x_2.$$

The set $\langle x_1, x_2 \rangle$ is called a conical segment.

Theorem 2. *Let an analytic operator F , monotone on the conical segment $\langle x_1, x_2 \rangle$, completely continuous, map $\langle x_1, x_2 \rangle$ into itself, i.e. $Fx_1 \geq x_1$ and $Fx_2 \leq x_2$, with $Fx_1 \neq x_1$ and $Fx_2 \neq x_2$. Let the derivative $F'(x)$ be a u_0 -positive operator for every x representable in the form $x = Fy$, where $y \in \langle x_1, x_2 \rangle$.*

Then the operator F has at least one asymptotically stable fixed point $x_0 \in \langle x_1, x_2 \rangle$.

In the proof of the theorem, the Schauder principle and Theorem 1 are used.

We pass to applications.

1. Let Ω be a bounded open domain of the n -dimensional space E_n of points $x = (x_1, \dots, x_n)$, belonging to the class $A^{(2,\lambda)}$. In the domain Ω there is considered

consider the quasilinear parabolic equation

$$M_1 u \equiv \frac{\partial u}{\partial t} - \sum_{i,k=1}^n a_{ik}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_k} + f\left(t, x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0, \quad (2)$$

where

$$\sum_{i,k=1}^n a_{ik}(t, x) \xi_i \xi_k \geq \gamma_0 \sum_{i=1}^n \xi_i^2 \quad (\gamma_0 > 0)$$

for all real ξ_i , $t, x \in [0, \omega] \times \Omega$. It is assumed that all the functions $a_{ik}(t, x)$, $\frac{\partial}{\partial x_j} a_{ik}(t, x)$, $f(t, x, u, v_1, \dots, v_n)$, and $\frac{\partial}{\partial x_j} f(t, x, u, v_1, \dots, v_n)$ satisfy, with respect to the variables $t, x \in [0, \omega] \times \bar{\Omega}$, a Hölder condition on each compact set with respect to $u, v_1, \dots, v_n$. Further, it is assumed that the function $f(t, x, u, v_1, \dots, v_n)$ is analytic in the variables $u, v_1, \dots, v_n$ and that the inequality

$$|f(t, x, u, v_1, \dots, v_n)| \leq C_1(R) + C_2(R) \sum_{i=1}^n |v_i|^2$$

holds, where $t, x \in [0, \omega] \times \bar{\Omega}$, $|u| \leq R$. Finally, it is assumed that the functions $a_{ik}(t, x)$ and $f(t, x, u, v_1, \dots, v_n)$ are ω -periodic in t .

Below, by a solution of equation (2) we shall mean any function $u(t, x)$ ($0 \leq t \leq T$, $x \in \bar{\Omega}$) that is continuous in the aggregate of variables $t, x \in [0, T] \times \bar{\Omega}$, that satisfies equation (2) for $t, x \in (0, T] \times \Omega$, and that, for $x \in \Gamma$, $0 \leq t \leq T$ (Γ is the boundary of the domain Ω), satisfies the zero boundary condition.

Let $u_0(t, x)$ ($-\infty < t < \infty$, $x \in \bar{\Omega}$) be some ω -periodic in t solution of equation (2). We shall call it Lyapunov stable if for every $\varepsilon > 0$ one can indicate a $\delta > 0$ such that:

- a) for any initial condition $u(0, x)$ satisfying the inequality

$$\|u(0, x) - u_0(0, x)\|_{C^{(1, \nu)}(\Omega)} \leq \delta \quad (0 < \nu < 1), \quad (3)$$

the Cauchy problem for equation (2) is globally solvable on $(0, \infty)$;

- b) from inequality (3) there follows the inequality

$$\|u(t, x) - u_0(t, x)\|_{C^{(1, \nu)}(\Omega)} \leq \varepsilon \quad (0 \leq t < \infty).$$

We shall call the solution $u_0(t, x)$ asymptotically Lyapunov stable if, in addition,

$$\lim \|u(t, x) - u_0(t, x)\|_{C^{(1, \nu)}(\Omega)} = 0 \quad (t \rightarrow \infty).$$

Let us introduce into consideration two functions $\psi_1(t, x)$ and $\psi_2(t, x)$. Below it is assumed that these functions satisfy, in the aggregate of variables $t, x \in [0, \omega] \times \bar{\Omega}$, some Hölder condition, are continuously differentiable in t , and twice continuously differentiable in the spatial variables in the domain $(0, \omega] \times \Omega$.

Theorem 3. Suppose there exist functions $\psi_1(t, x)$ and $\psi_2(t, x)$ ($\psi_1(t, x) \leq \psi_2(t, x)$ for $t, x \in [0, \omega] \times \bar{\Omega}$) satisfying the inequalities

$$\begin{aligned} M_1 \psi_1(t, x) &\leq 0 \leq M_1 \psi_2(t, x) & (0 < t \leq \omega, x \in \Omega), \\ \psi_1(t, x) &\leq 0 \leq \psi_2(t, x) & (0 \leq t \leq \omega, x \in \Gamma), \\ \psi_1(0, x) &\leq \psi_1(\omega, x), \quad \psi_2(0, x) \geq \psi_2(\omega, x) & (x \in \Omega). \end{aligned}$$

Suppose that neither of the functions $\psi_1(0, x)$ and $\psi_2(0, x)$ is the initial condition of an ω -periodic solution of equation (2).

Then equation (2) has at least one Lyapunov-asymptotically stable ω -periodic in t solution $u_0(t, x)$, satisfying the inequalities

$$\psi_1(t, x) \leq u_0(t, x) \leq \psi_2(t, x) \quad (0 \leq t \leq \omega, x \in \bar{\Omega}).$$

2. Consider the ordinary differential equation of second order

$$M_2x \equiv \ddot{x} + f(t, x, \dot{x}) = 0. \quad (4)$$

We shall assume that the function $f(t, x, y)$, continuous in the aggregate of variables, is ω -periodic in t , analytic in the variables x and y , and satisfies the inequality

$$|f(t, x, y)| \leq C_1(R) + C_2(R)|y|^2 \quad (0 \leq t \leq \omega, |x| \leq R).$$

Theorem 4. Let there exist twice continuously differentiable functions $x_1(t)$ and $x_2(t)$ ($x_1(t) \leq x_2(t)$ for $0 \leq t \leq \omega$), satisfying the inequalities

$$M_2x_1(t) \geq 0 \geq M_2x_2(t) \quad (0 \leq t \leq \omega),$$

$$x_1(0) = x_1(\omega), \quad \dot{x}_1(0) \leq \dot{x}_1(\omega), \quad x_2(0) = x_2(\omega), \quad \dot{x}_2(0) \geq \dot{x}_2(\omega).$$

Let neither of the vectors $\{x_1(0), \dot{x}_1(0)\}$ and $\{x_2(0), \dot{x}_2(0)\}$ be an initial condition of an ω -periodic solution of equation (4). Then equation (4) has at least one ω -periodic solution $x_0(t)$, unstable in both directions of time, satisfying the inequalities

$$x_1(t) \leq x_0(t) \leq x_2(t) \quad (0 \leq t \leq \omega).$$

Theorem 4 complements the results of [2].

3. As a final example, consider the problem of the existence of stable periodic solutions for the equation of order m

$$M_3x \equiv x^{(m)} + a_1(t)x^{(m-1)} + \dots + a_m(t)x = f(t, x), \quad (5)$$

where the continuous functions $a_i(t)$ and $f(t, x)$ are ω -periodic in t . It is assumed below that the function $f(t, x)$ is analytic in the variable x and is nondecreasing in this variable. In addition, it is assumed that the multipliers of the equation $M_3x = 0$ lie inside the unit circle and that the Cauchy function $K(t, \tau)$ of this equation is nonnegative for all $-\infty < \tau \leq t < \infty$.

Theorem 5. Let there exist m -times continuously differentiable ω -periodic functions $x_1(t)$ and $x_2(t)$ ($x_1(t) \leq x_2(t)$ for $0 \leq t \leq \omega$), such that the inequalities

$$M_3x_1(t) \leq f[t, x_1(t)], \quad M_3x_2(t) \geq f[t, x_2(t)] \quad (0 \leq t \leq \omega)$$

are satisfied. Let neither of the vectors $\{x_1(0), \dots, x_1^{(m-1)}(0)\}$ and $\{x_2(0), \dots, x_2^{(m-1)}(0)\}$ be an initial condition of an ω -periodic solution of equation (5).

Then equation (5) has at least one Lyapunov-asymptotically stable ω -periodic solution $x_0(t)$, satisfying the inequalities

$$x_1(t) \leq x_0(t) \leq x_2(t) \quad (0 \leq t \leq \omega).$$

For $m = 2$, the restrictions on the linear part of equation (5) can be substantially weakened: it is sufficient to assume that the Cauchy function $K(t, \tau)$ is nonnegative for all $\tau \leq t \leq \tau + \omega$ and $0 \leq \tau \leq \omega$.

Voronezh State University

Received

5 III 1969

REFERENCES

1. M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, 1962.
2. H. W. Knobloch, *Math. Zs.*, **82**, 177 (1963).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.