

ON THE CENTRAL LIMIT THEOREM

M. I. Gordin

1969

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196901.82106>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.217

MATHEMATICS

M. I. Gordin

ON THE CENTRAL LIMIT THEOREM FOR STATIONARY PROCESSES

(Presented by Academician Yu. V. Linnik on 26 II 1969)

In the present note it is shown that the central limit theorem (c.l.t.) for certain classes of stationary, in the narrow sense, random processes can be obtained from the c.l.t. for stationary sequences of martingale differences.

Suppose that a space X with a σ -algebra of sets M and a probability measure P is given. The spaces $\mathbf{L}_p$ correspond to the measure P ; $|f|_p$ is the norm of the function f in $\mathbf{L}_p$.

If a σ -algebra L is contained in M , then $H(L)$ denotes the Hilbert space of those functions from $\mathbf{L}_2$ that are measurable with respect to L .

The symbol P_G denotes the orthoprojector onto the closed subspace $G \subset H = \mathbf{L}_2$.

Let T be an ergodic automorphism of the space X with measure P , and let M_0 be a σ -algebra such that $T^{-1}(M_0) \subset M_0$. The relation $Uf(x) = f(Tx)$ defines a unitary operator in H .

Finally, set $M_k = T^{-k}(M_0)$, $H_k = H(M_k) = U^k H_0$, $S_k = H_k \ominus H_{k+1}$. By R we denote the linear space of those elements $g \in H$ such that $g \in H_k \ominus H_l$ for some k and l , $-\infty < k \leq l < \infty$.

Theorem 1. Let $f \in \mathbf{L}_2$ and

$$\inf_{g \in R} \lim_{n \rightarrow \infty} \frac{\left| \sum_{k=0}^{n-1} U^k(f - g) \right|_2}{\sqrt{n}} = 0.$$

Then there exists

$$\lim_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} U^k f \right|_2 / \sqrt{n} = \sigma \quad (1)$$

and

$$\mathbf{P} \left\{ \sum_{k=0}^{n-1} U^k f / \sqrt{n} < z \right\} \xrightarrow{n \rightarrow \infty} \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^z e^{-u^2/2\sigma^2} du. \quad (2)$$

We outline the proof of the theorem. Let $\varepsilon_p > 0$, $\varepsilon_p \xrightarrow{p \rightarrow \infty} 0$, $f_p \in R$, and

$$\overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} U^k (f - f_p) \right|_2 / \sqrt{n} < \varepsilon_p.$$

Consider the chain of equalities

$$\begin{aligned} f &= f_p + f - f_p = \sum_{l=-\infty}^{\infty} P_{S_l} f_p + f - f_p = \sum_{l=-\infty}^{\infty} U^{-l} P_{S_l} f_p + \\ &+ \sum_{l=-\infty}^{\infty} \sum_{m=0}^{-l-1} U^m P_{S_l} f_p - U \sum_{l=0}^{\infty} \sum_{m=0}^{-l-1} U^m P_{S_l} f_p + f - f_p = h_p + g_p - U g_p + f - f_p. \end{aligned}$$

Further,

$$\begin{aligned} &\overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} U^k (f - h_p) \right|_2 / \sqrt{n} \leq \\ &\leq \overline{\lim}_{n \rightarrow \infty} \left\{ |g_p - U^n g_p|_2 + \left| \sum_{k=0}^{n-1} U^k (f - f_p) \right|_2 \right\} / \sqrt{n} < \varepsilon_p. \end{aligned} \quad (3)$$

Let us now note that

$$h_p = \sum_{l=-\infty}^{\infty} U^{-l} P_{S_l} f_p \in S_0, \quad U^k h_p \in S_k.$$

Hence it follows that $U^k h_p$ is measurable with respect to M_k and is orthogonal to $H_{k+1} = H(M_{k+1})$, i.e., the sequence $U^{-k} h_p$ is an ergodic sequence of martingale differences. Therefore (see ^(1, 2)) the quantities

$$\sum_{k=0}^{n-1} U^k h_p / \sqrt{n}$$

are asymptotically normally distributed with variance $\sigma_p^2 = |h_p|^2$.

The sequence σ_p converges to some limit σ , since

$$\begin{aligned} |\sigma_p - \sigma_{p'}| &\leq |h_p - h_{p'}|_2 = \left| \sum_{k=0}^{n-1} U^k(h_p - h_{p'}) \right|_2 / \sqrt{n} \leq \\ &\leq \overline{\lim}_{n \rightarrow \infty} \left\{ \left| \sum_{k=0}^{n-1} U^k(f - h_p) \right|_2 + \left| \sum_{k=0}^{n-1} U^k(f - h_{p'}) \right|_2 \right\} / \sqrt{n} \leq \varepsilon_p + \varepsilon_{p'}. \end{aligned}$$

Relation (2) now follows from Lemma 5.3 of ⁽⁴⁾; equality (1) follows from (3) and from the fact that $\sigma_p \rightarrow \sigma$.

Theorem 2. *Let T be an ergodic automorphism, M_0 the same σ -algebra as in Theorem 1; $f \in \mathbf{L}_{2+\delta}$ for some $0 \leq \delta \leq \infty$ and*

$$\sum_{A \geq 0} (|P_{H_A} f|_{(2-\delta)/(1+\delta)} + |f - P_{H_{-A}} f|_{(2+\delta)/(1+\delta)}) < \infty.$$

Then the condition of Theorem 1 is satisfied.

Let us explain how Theorem 2 is proved. Instead of $\int_X g(x)h(x)\mathbf{P}(dx)$ we shall write (g, h) . Put

$$f_1^{(A)} = P_{H_A} f, \quad f_2^{(A)} = f - P_{H_{-A}} f, \quad r_i^{(A)}(k) = (f_i^{(A)}, U^k f_i^{(A)}) \quad (i = 1, 2).$$

From Hölder' s inequality and the well-known property of conditional mathematical expectations (see, for example, ⁽³⁾, p. 508) it follows that

$$\begin{aligned} |r_i^{(A)}(k)| &= |(f_i^{(A)}, U^{|k|} f_i^{(A)})| = |(f_i^{(A)}, U^{\pm|k|} f_i^{(A+|k|)})| \leq \\ &\leq |f_i^{(A)}|_{2+\delta} |f_i^{(A+|k|)}|_{(2-\delta)/(1+\delta)} \leq 2|f|_{2+\delta} |f_i^{(A+|k|)}|_{(2+\delta)/(1+\delta)}. \end{aligned}$$

Using this estimate, we find that

$$\begin{aligned} &\overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=0}^{n-1} U^k(f_1^{(A)} + f_2^{(A)}) \right|_2 / \sqrt{n} \leq \sum_{i=1}^2 \overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=1}^{n-1} U^k f_i^{(A)} \right|_2 / \sqrt{n} = \\ &= \sum_{i=1}^2 \lim_{n \rightarrow \infty} \left(\sum_{|k| < n} \left(1 - \frac{|k|}{n} \right) r_i^{(A)}(k) \right)^{1/2} \leq 2|f|_{2+\delta}^{1/2} \sum_{i=1}^2 \left(\sum_{k=A}^{\infty} |f_i^{(k)}|_{(2+\delta)/(1+\delta)} \right)^{1/2} \rightarrow 0, \quad A \rightarrow \infty. \end{aligned}$$

It remains to note that

$$f - f_1^{(A)} - f_2^{(A)} \in R.$$

Remark 1. Theorems 1 and 2 can be reformulated for the case where T is an endomorphism. In this case the spaces S_k form a sequence infinite in one direction.

Remark 2. Under the conditions of Theorem 2, the equality $f = g + Uh - h$ holds, where $g, h \in \mathbf{L}^{(2+\delta)/(1+\delta)}$, g is measurable with respect to M_0 and has zero integrals over all sets from M_1 . Such a representation for $\delta = 0$ is useful in the proof of more refined limit theorems, for example, in the proof of weak convergence of distributions in the space of continuous functions corresponding to the sequence of random polygonal lines constructed from the sums

$$\sum_{k=0}^{n-1} U^k f$$

to a measure corresponding to Brownian motion.

Remark 3. From Theorem 2 it is not difficult to obtain a number of theorems concerning processes with strong and uniformly strong mixing and functionals of such processes.

In particular, Theorems 18.6.1, 18.6.2, and 18.6.3 from the book ⁽³⁾ are consequences of Theorem 2. Moreover, under the conditions of Theorem 18.6.1 and with a certain strengthening of the conditions of Theorems 18.6.2 and 18.6.3 from ⁽³⁾, one can obtain the representation $f = g + Uh - h$, $g \in S_0$, $h \in \mathbf{L}_2$.

The author expresses gratitude to I. A. Ibragimov for his attention to the work.

Leningrad State University
named after A. A. Zhdanov

Received
23 I 1969

REFERENCES

- ¹ R. Billingsley, Proc. Am. Math. Soc., **12**, 5, 788 (1961).
- ² I. A. Ibragimov, Theory of Probability and Its Applications, **8**, 1, 89 (1963).
- ³ I. A. Ibragimov, Yu. V. Linnik, *Independent and Stationarily Related Random Variables*, Moscow, 1965.
- ⁴ V. P. Leonov, *Some Applications of Higher Seminvariants to the Theory of Stationary Random Processes*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.