

ON FUNCTORS DEFINED BY REFLEXIVE $\backslash(K\backslash)$ -SPACES

V. A. Geiler

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.81721>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.88:513.83

MATHEMATICS

V. A. GEILER

ON FUNCTORS DEFINED BY REFLEXIVE K -SPACES

(Presented by Academician L. V. Kantorovich on 24 I 1969)

Functors on the category of locally convex spaces were studied by G. H. Berman^(1,2). He constructed a duality theory for functors on the category $\mathcal{FN}$ of nuclear Fréchet spaces. In particular, he studied the functor $\Lambda_{\mathfrak{M}}$, defined by the perfect space of sequences λ , which, from the point of view of the theory of K -spaces, is a discrete reflexive K -space of countable type^(3,4). In the present note, for each reflexive K -space X , endowed with a certain locally convex topology, a functor $\Phi_X^{\mathfrak{M}}$ is constructed, of which $\Lambda_{\mathfrak{M}}$ is a special case. We shall also consider a certain functor $\Omega_A^{\mathfrak{S}}$, and it turns out that on certain categories (containing $\mathcal{FN}$) functors of this type are isomorphic to Ω_A , Σ_A , $\Phi_X^{\mathfrak{M}}$, and also to the ε_A introduced in⁽⁵⁾.

In what follows X denotes a reflexive K -space, $\bar{X}$ its conjugate, $\mathfrak{M}$ a certain system of bounded absolutely convex normal subsets of $\bar{X}$ covering $\bar{X}$ (and called a topologizing system for X); if all the sets in $\mathfrak{M}$ are $\sigma(\bar{X}, X)$ -bicomcompact, then $\mathfrak{M}$ is called bicomcompact. By $X_{\mathfrak{M}}$ we denote the space X , endowed with the locally convex topology generated by the polars of the sets in $\mathfrak{M}$. Using the results of⁽⁶⁾, it is not difficult to show that among the spaces obtained in this way there is X_{τ} —the space X endowed with the Mackey topology $\tau(X, \bar{X})$; we also note that a subset of $X_{\mathfrak{M}}$ is bounded if and only if it belongs to the system $\mathfrak{B}$ of $\sigma(X, \bar{X})$ -bounded subsets of X . Using the results of A. G. Pinsker⁽⁴⁾, Chap. IV, Theorems 1.31, 4.13), we realize X as the base of a K -space $L_{\text{loc}}^1(T; \mu)$, where T is a locally bicomcompact space which is a sum of extremally disconnected bicompacts; μ is a positive measure on T such that the families of locally μ -negligible⁽⁷⁾ and nowhere dense subsets of T coincide, and $L_{\text{loc}}^1(T; \mu)$ is the quotient space of the space $\mathcal{L}_{\text{loc}}^1(T; \mu)$ of locally summable functions by the subspace of locally μ -negligible functions. By θ we shall denote the canonical mapping of $\mathcal{L}_{\text{loc}}^1(T; \mu)$ onto $L_{\text{loc}}^1(T; \mu)$.

Let E denote a separable locally convex space, and $\mathfrak{U}$ the family of its absolutely convex neighborhoods of zero. By $\mathcal{X}_{\mathfrak{M}}(E)$ (respectively $\mathcal{X}_{\mathfrak{M}}[E]$) we denote the space of μ -measurable functions from T into E such that for every continuous seminorm p on E , $\theta(pf) \in X$ (for every $a' \in E'$, $\theta(a'f) \in X$), endowed with the topology generated by the seminorms

$$f \mapsto p_{M^0}(\theta(p_U f))$$

(respectively

$$f \mapsto \sup_{a' \in U^0} p_{M^0}(\theta(a' f)),$$

where $M \in \mathfrak{M}$, $U \in \mathfrak{U}$. These spaces do not depend on the choice of the realization of X . By $X_{\mathfrak{M}}(E)$ (respectively $X_{\mathfrak{M}}[E]$) we denote the separable locally convex space associated with $\mathcal{X}_{\mathfrak{M}}(E)$ ($\mathcal{X}_{\mathfrak{M}}[E]$).

Theorem 1. If E is a nuclear space, then $X_{\mathfrak{M}}(E)$ is isomorphic to $X_{\mathfrak{M}}[E]$.

If E, F are locally convex spaces and $\mathfrak{S}$ is a system of bounded subsets of E , then $L^{\mathfrak{S}}(E, F)$ is the space of continuous li-

linear maps from E to F , endowed with the $\mathfrak{S}$ -topology; by $\mathfrak{B}$ we shall denote the system of all bounded subsets of E , and by $\mathfrak{E}$ that of all equicontinuous subsets of E' .

Theorem 2. Let E be a complete nuclear space, and let $\mathfrak{M}$ be a bicomcompact topologizing system for $\bar{X}$.

Each of the following conditions is sufficient for $X_{\mathfrak{M}}(E)$ to be isomorphic to $L_{\mathfrak{M}}(X_{\mathfrak{M}}, E)$: a) E is dually nuclear ⁽⁸⁾; b) let $\mathfrak{a}$ be the character of E ; if $\mathfrak{F}$ is a collection of nowhere dense subsets T and $\text{card}(\mathfrak{F}) \leq \mathfrak{a}$, then $\bigcup \mathfrak{F}$ is nowhere dense.

Corollary. If, under the hypotheses of Theorem 2, the system $\mathfrak{M}$ is bicomcompact, then

$$X_{\mathfrak{M}}(E) \simeq X_{\mathfrak{M}} \hat{\otimes} E.$$

In what follows, $\mathcal{L}$ denotes the category of separated locally convex spaces, $\mathcal{EL}$ the full subcategory of complete spaces in $\mathcal{L}$, and $\mathcal{IT}$ the category of infrabarreled spaces; we shall also consider the following full subcategories of $\mathcal{EL}$: $\mathcal{BLT}$, of barrelled spaces; $\mathcal{N}$, of nuclear spaces; $\mathcal{DN}$, of nuclear dually nuclear spaces; $\mathcal{RN}$, of reflexive nuclear spaces; and $\mathcal{RDN}$, of reflexive spaces in $\mathcal{DN}$.

For each space $X_{\mathfrak{M}}$ we define a functor $\Phi_X^{\mathfrak{M}}$ on $\mathcal{L}$ as follows: $\Phi_X^{\mathfrak{M}} E = X_{\mathfrak{M}}(E)$, and for $\alpha \in L(E, F)$, $\Phi_X^{\mathfrak{M}} \alpha$ is the mapping induced by passage to quotient spaces from the mapping $f \mapsto \alpha f$ from $\mathcal{X}_{\mathfrak{M}}(E)$ to $\mathcal{X}_{\mathfrak{M}}(F)$. The functors Ω_A and Σ_A are the same as in (1). Let $A \in \mathcal{L}$; $\mathfrak{S}$ is a system of bounded subsets of A covering A ; define the functor $\Omega_A^{\mathfrak{S}}$ on the category $\mathcal{K}$ by

$$\Omega_A^{\mathfrak{S}} E = L_{\mathfrak{S}}(A, E), \quad \Omega_A^{\mathfrak{S}} \alpha = \Omega_A \alpha \quad (\alpha \in L(E, F)).$$

The functor $\Omega_{A_c}^{\mathfrak{E}}$ will also be denoted by Ψ_A .

The following assertions are valid:

- a) if $A \in \mathcal{BL}$, then Σ_A is isomorphic to Ψ_A on $\mathcal{N}$; if $A \in \mathcal{N}$, then Σ_A is isomorphic to Ψ_A on $\mathcal{BL}$;
- b) for every $A \in \mathcal{L}$, ε_A is isomorphic to $\Omega_{A_c}^{\mathfrak{e}}$ on $\mathcal{L}$;
- c) on $\mathcal{DN}$, $\Phi_X^{\mathfrak{M}}$ is isomorphic to $\Omega_{X_\tau}^{\mathfrak{M}}$.

Proof. Assertion a) follows from Grothendieck's theorem (⁹, Ch. II, Theorem 6); b) from Corollary 2 to Proposition 4 (¹⁰); c) from Theorem 2.

If R and S are such functors that the class of mappings from R to S is a set, then by $H_b(R, S)$ we shall denote the space of mappings from R to S , endowed with the initial topology with respect to the family of mappings

$$P_E : H(R, S) \rightarrow L_{\mathfrak{B}}(RE, SE). \quad (1)$$

A functor R is called continuous on the category $\mathcal{K}$ if, for every pair of spaces E, F from $\mathcal{K}$, the mapping

$$R : L_{\mathfrak{B}}(E, F) \rightarrow L_{\mathfrak{B}}(RE, RF)$$

is continuous; R is called regular on $\mathcal{K}$ if $H_b(\Omega_A, R)$ is canonically isomorphic to the space RA for every A in $\mathcal{K}$ (¹).

Theorem 3. For every $A \in \mathcal{L}$, the functor $\Omega_A^{\mathfrak{S}}$ is continuous on $\mathcal{L}$ and regular on $F\mathcal{T}$.

Theorem 4. Let $A, B \in \mathcal{L}$; let $\mathfrak{S}$ (respectively $\mathfrak{T}$) be a system of absolutely convex weakly bicomact subsets of A (respectively of B).

If for every $u \in L(A'_{\mathfrak{S}}, B'_{\mathfrak{T}})$ the adjoint mapping $u' \in L(B, A)$ (in particular, if B is a Mackey space), then

$$H_b(\Omega_A^{\mathfrak{S}}, \Omega_B^{\mathfrak{T}})$$

is isomorphic to

$$L_{\mathfrak{B}}(A'_{\mathfrak{S}}, B'_{\mathfrak{T}}).$$

By $\mathcal{DS}$ is denoted the functor dual (¹) to S .

Theorem 5. Let $A \in \mathcal{BL}$, and let the sets in $\mathfrak{S}$ be relatively weakly bicomact. On the category $\mathcal{N}$ we have: a) $\mathcal{D}\Omega_A^{\mathfrak{S}} = \Omega_{A'_c}^{\mathfrak{S}}$; b) if A_τ is barrelled, then

$$\mathcal{D}^2\Omega_A^{\mathfrak{S}} = \Omega_{A_\tau}^{\mathfrak{S}};$$

c) if A is reflexive, then Ω_A is reflexive.

Let us consider some consequences of these theorems.

Corollary 1. *Let X, Y be reflexive K -spaces, and let $\mathfrak{M}$ (co-

respectively $\mathfrak{N}$) is a topologizing system for X (respectively Y); the functors $\Phi_X^{\mathfrak{M}}$ and $\Phi_Y^{\mathfrak{N}}$ are defined on $\mathcal{DN}$.

Then: a) $\Phi_X^{\mathfrak{M}}$ is continuous on $\mathcal{DN}$ and regular on $\mathcal{FDM}$; b) if the systems $\mathfrak{M}$ and $\mathfrak{N}$ are bicomact, then $H_b(\Phi_X^{\mathfrak{M}}, \Phi_Y^{\mathfrak{N}})$ is isomorphic to $L_{\mathfrak{B}}(X_{\mathfrak{M}}, Y_{\mathfrak{N}})$; c) if $\mathfrak{M}$ is bicomact, then $\mathcal{D}\Phi_X^{\mathfrak{M}} = \Phi_X^{\mathfrak{B}}$; if $X_{\mathfrak{M}}$ is semireflexive, then $\mathcal{D}^2\Phi_X^{\mathfrak{M}} = \Phi_X^{\mathfrak{B}}$; if $X_{\mathfrak{M}}$ is a reflexive locally convex space, then $\Phi_X^{\mathfrak{M}}$ is reflexive.

Corollary 2. Let $A, B \in \mathcal{CL}$ (respectively $\mathcal{N}$); the functors $\Sigma_A, \Sigma_B, \Omega_A$ are defined on $\mathcal{N}$ (respectively $\mathcal{CL}$).

Then: a) Σ_A is continuous on $\mathcal{N}$ (respectively $\mathcal{CL}$) and regular on $\mathcal{RN}$ (respectively $\mathcal{LT}$); Ω_A is regular on $\mathcal{LT}$; b) $H_b(\Sigma_A, \Sigma_B)$ is isomorphic to $L_{\mathfrak{B}}(A, B)$; c) $\mathcal{D}\Omega_A = \Sigma_A$ on $\mathcal{N}$ (respectively $\mathcal{CL}$); $\mathcal{D}\Sigma_A = \Omega_A$ on $\mathcal{RN}$ (respectively $\mathcal{LT}$); d) Σ_A and Ω_A are reflexive on $\mathcal{RN}$ (respectively $\mathcal{LT}$); if A is reflexive, then Σ_A is reflexive on $\mathcal{N}$.

Using Theorem 2.5 of paper ⁽²⁾, we obtain

Corollary 3. For every functor defined on $\mathcal{N}$, there exists a continuous dual functor defined on the category of complete spaces satisfying the approximation condition.

Remark 1. Along with the definition of a dual functor given in (1), it is natural also to consider the following definition. Let S be a functor from $\mathcal{K}$ to $\mathcal{L}$; by ∂S denote the functor acting as follows: for $E \in \mathcal{L}$,

$$\partial SE = H_b(S, \Psi_E);$$

if $u \in L(E, F)$, then

$$((\partial Su)\alpha)_X = (\Psi_{Xu})\alpha_X,$$

where $\alpha \in H_b(S, \Psi_E)$,

$$\alpha_X \in L_{\mathfrak{B}}(SX, L_{\varepsilon}(E'_c, X)) \simeq L_{\mathfrak{B}}(SX, L_{\varepsilon}(X'_c, E')),$$

$$(\Psi_{Xu})\alpha_X \in L_{\mathfrak{B}}(SX, L_{\varepsilon}(X'_c, F')) \simeq L_{\mathfrak{B}}(SX, L_{\varepsilon}(F'_c, X)) = L_{\mathfrak{B}}(SX, \Psi_{FX}).$$

With such a definition of duality, the notion of reflexivity of functors naturally arises, and it turns out that the functors Ω_A and Ψ_A are dual to one another and are reflexive on the category $\mathcal{LT}$ in this sense.

Remark 2. We indicate one more generalization of the results of ^(1,2,5). We shall call a functor t from a category $\mathcal{K} \subset \mathcal{L}$ to the category of partially ordered sets O topologizing if tE , for every E in $\mathcal{K}$, is a saturated system of absolutely convex bounded subsets of E , and if for $\alpha \in L(E, F)$ there is a canonical distribution α on tE . Examples of topologizing functors on $\mathcal{L}$: b , bE is the collection of all bounded subsets of E ; p , pE is the collection of all precompact subsets of E ; c , cE is the saturated system generated by the absolutely convex bicomact subsets of E ; s , sE is the collection of finite-dimensional absolutely convex subsets of E . The pair $\langle \mathcal{K}, t \rangle$ will be called a topologized category. A functor $S : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ will be called continuous from $\langle \mathcal{K}_1, t \rangle$ to $\langle \mathcal{K}_2, u \rangle$ if all the induced mappings $L_{tE}(E, F) \rightarrow L_{uSE}(SE, SF)$ are continuous. To functors in topologized categories one can extend the notions of regularity, duality, etc.

In conclusion I consider it my pleasant duty to express my deep gratitude to A. G. Pinsker and D. A. Raikov for their great attention to this work.

Mordovian State University

Received
18 I 1969

REFERENCES

1. G. Kh. Berman, DAN, **154**, No. 3, 497 (1964).
2. G. Kh. Berman, Matem. sborn., **74**, No. 4, 475 (1967).
3. B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, Moscow, 1961.
4. L. V. Kantorovich, B. Z. Vulikh, A. G. Pinsker, *Functional Analysis in Partially Ordered Spaces*, Moscow-Leningrad, 1950.
5. N. Popa, Rev. roum. math. pur. appl., **13**, No. 5, 669 (1968).
6. J. Amemia, Proc. Internat. Sympos. Linear Spaces, Jerusalem, 1960, p. 14.
7. N. Bourbaki, *Integration*, Moscow, 1967.
8. A. Pietsch, *Nuclear Locally Convex Spaces*, Moscow, 1967.
9. A. Grothendieck, Mem. Am. Math. Soc., No. 16 (1955).
10. L. Schwartz, Ann. Inst. Fourier, **7**, 1 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.