

LIE SUBGROUPS OF THE LORENTZ GROUP WITH A FINITE GROUP OF COMPONENTS

A. N. Golubyatnikov

1969

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196901.80252>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 512.865+530.12+531.18

MATHEMATICS

A. N. GOLUBYATNIKOV

LIE SUBGROUPS OF THE LORENTZ GROUP WITH A FINITE GROUP OF COMPONENTS

(Presented by Academician L. I. Sedov on 18 X 1968)

Consider the Lorentz group G of linear transformations of the variables x^i ($i = 1, 2, 3, 4$) that leave invariant the quadratic form $x^{12} + x^{22} + x^{32} - x^{42}$. The infinitesimal operators of its connected component of the identity, called the proper Lorentz group H_0 , have the form

$$\begin{aligned} X_1 &= x^2 \partial_3 - x^3 \partial_2, & X_2 &= x^3 \partial_1 - x^1 \partial_3, & X_3 &= x^1 \partial_2 - x^2 \partial_1, \\ X_4 &= x^1 \partial_4 + x^4 \partial_1, & X_5 &= x^2 \partial_4 + x^4 \partial_2, & X_6 &= x^3 \partial_4 + x^4 \partial_3, \end{aligned}$$

where $\partial_i = \partial/\partial x^i$.

To study subgroups of the group G , consider its spinor representation by the group A of matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}, \quad \begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & -\bar{\alpha} \end{pmatrix},$$

where the group H_0 is put into correspondence with the group of complex unimodular second-order matrices α according to the law

$$c' = \alpha c \alpha^*, \quad c = \begin{pmatrix} x^4 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^4 - x^3 \end{pmatrix}.$$

The bar denotes complex conjugation, and the asterisk Hermitian conjugation. In this representation, to the reflection of x^α ($\alpha = 1, 2, 3$) there correspond the matrices

$$\pm \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}, \quad \text{where } \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and to the reflection of x^4 , the matrices

$$\pm \begin{vmatrix} 0 & \tau \\ -\tau & 0 \end{vmatrix}.$$

Since this representation is two-valued, we shall consider only subgroups of the group A containing the matrices $\pm e$ (e is the identity matrix), corresponding to subgroups of the group G .

All finite subgroups of the group G are conjugate in it to finite subgroups of its orthogonal subgroup containing orthogonal transformations of the variables x^α and the reflection of x^4 , which are known ^(1, 2), since a transformation v , under which the matrices a_k of the elements m_k ($k = 1, \dots, r$) of a finite subgroup are reduced to unitary form corresponding to an orthogonal subgroup of the group G , determined by the relation

$$v^* v = \frac{1}{r} \sum_{k=1}^r m_k^* m_k$$

⁽³⁾, is reduced to a transformation belonging to the identity component of the group A , i.e., corresponding to a proper Lorentz transformation from H_0 .

The classification of real Lie subalgebras of the Lie algebra of the Lorentz group is known ^(4, 5). From it it is not difficult to reconstruct, up to conjugacy in the group G , all connected subgroups H_μ ($\mu = 0, 1, \dots, 13$). To find Lie subgroups of the group G of positive dimension with a finite group of components, since the identity component of a Lie group is its invariant subgroup ⁽⁶⁾, we consider, within the spinor representation, the normalizers N_μ , $bab^{-1} = a'$, $a, a' \in H_\mu$, $b \in N_\mu$, of these connected subgroups in the group G , which itself is a normalizer for H_0 .

In this case, for the groups H_μ , the matrices α of which have the form $\begin{vmatrix} x & y \\ 0 & 1/x \end{vmatrix}$, $x \neq 0$, the following cases are possible:

1. y is an arbitrary complex number; in this case the matrices α corresponding to the normalizer have the form $\begin{vmatrix} p & q \\ 0 & 1/p \end{vmatrix}$, $p \neq 0$, where p and q are arbitrary complex numbers;
2. $\text{Im } y = 0$; for $\text{Im } x = 0$ we obtain, for the normalizer, matrices α of the form $\begin{vmatrix} i\rho & i\sigma \\ 0 & -i/\rho \end{vmatrix}$ or $\begin{vmatrix} \rho & \sigma \\ 0 & 1/\rho \end{vmatrix}$, where $\text{Im } \rho = \text{Im } \sigma = 0$; for $x = \pm 1$, of the form $\begin{vmatrix} i\rho & q \\ 0 & -i/\rho \end{vmatrix}$ or $\begin{vmatrix} \rho & q \\ 0 & 1/\rho \end{vmatrix}$, where $\text{Im } \rho = 0$;
3. $y = 0$ gives the same matrices of the form $\begin{vmatrix} p & 0 \\ 0 & 1/p \end{vmatrix}$ or $\begin{vmatrix} 0 & q \\ -1/q & 0 \end{vmatrix}$.

The first case corresponds to the groups $H_1, H_2(|x| = 1), H_3(\text{Im } x = 0), H_4(x = \pm 1)$, and $H_5(x = \pm e^{(k+i)\varphi}, \text{Im } \varphi = 0, k > 0)$; the second to $H_6(\text{Im } x = 0)$ and $H_7(x = \pm 1)$; the third to $H_8, H_9(|x| = 1), H_{10}(x = \pm e^{(k+i)\varphi}, \text{Im } \varphi = 0, k > 0)$, and $H_{11}(\text{Im } x = 0)$.

The groups $H_{12} \left(\alpha = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \right)$ and $H_{13} \left(\alpha = \begin{pmatrix} x & y \\ \bar{y} & \bar{x} \end{pmatrix} \right)$ give normalizers whose matrices α have, respectively, the form $\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}$ and $\begin{pmatrix} x & y \\ \bar{y} & \bar{x} \end{pmatrix}$, or $\begin{pmatrix} x & y \\ -\bar{y} & -\bar{x} \end{pmatrix}$.

If the matrix $\bar{a}$ does not belong to H_μ simultaneously with a , then the matrices of the normalizer N_μ have the form $\begin{pmatrix} a & 0 \\ 0 & \pm \bar{a} \end{pmatrix}$; if $\bar{a}$ belongs to H_μ , then they have the form $\begin{pmatrix} a & 0 \\ 0 & \pm \bar{a} \end{pmatrix}$ or $\begin{pmatrix} 0 & a \\ \pm \bar{a} & 0 \end{pmatrix}$. For the groups H_5 and H_{10} , $\bar{a}$ does not belong to the group.

All the normalizers N_μ found, for each of their subgroups with a finite group of components, make it possible to choose representatives of conjugacy classes modulo H_μ so that they form a finite group isomorphic to the quotient group modulo H_μ . Therefore, for the classification up to conjugacy of subgroups of the group G with a finite group of components, it suffices, for a fixed group H_μ , to list all finite subgroups of N_μ not conjugate in it, whose intersection with H_μ is equal to $\pm e$, the element corresponding to the identity of the group G . All of them will give distinct non-conjugate subgroups of G with a finite group of components.

Taking into account the results obtained and the explicit form of the matrices of the finite groups in the spinor representation, we shall write out all possible finite groups isomorphic to the component groups of subgroups of the group G . In parentheses are indicated the infinitesimal operators, and the notation of finite groups according to Shubnikov [1]* is used. For $H_0, H_1(X_3, X_6, X_4 - X_2, X_1 + X_5), H_2(X_3, X_4 - X_2, X_1 + X_5), H_{12}(X_1, X_2, X_3)$ we have 1, 1 : 2, $\bar{2}$, 1· m , $\bar{2} \cdot m$; for $H_3(X_6, X_4 - X_2, X_1 + X_5), H_4(X_4 - X_2, X_1 + X_5)$ n , $n : 2, \bar{2}n \cdot m, n \cdot m, m \cdot n : m, 2n \cdot m$ ($n = 1, 2, \dots$);

for $H_5(X_3 + kX_6, X_4 - X_2, X_1 + X_5)$ $n, \bar{2}n, n : m$ ($n = 1, 2, \dots, k > 0$)

for $H_6(X_6, X_4 - X_2), H_7(X_4 - X_2)$ 1, 2, 1 : 2, $\bar{2}$, 2 : 2, 1 : m , 2 : m , 1· m , 2· m , $m \cdot 1 : m$, $m \cdot 2 : m$, $\bar{2} \cdot m$, 1 : $2'$, 1· m' , $m' \cdot 1 : m$, $\bar{2} \cdot m'$;

for $H_8(X_3, X_6), H_9(X_3), H_{13}(X_3, X_4, X_5)$ 1, $\bar{1}$, 1 : 2, 1 : $\bar{2}$, 1 : 2, $\bar{2}$, $\bar{2} \times \bar{1}$, 1· m , 1· $\underline{m}$, $\bar{1} \cdot m$, $\bar{2} \cdot m$, 2· $\underline{m}$, $\bar{2} \cdot m$, $\bar{2} \cdot \underline{m}$, $\bar{2} \cdot m \times \bar{1}$;

* n denotes a rotation about the axis x^3 through the angle $2\pi/n$, 1 : 2 —about the axis x^2 through π , $\cdot m$ and an overbar denote reflection of x^3 , $\cdot m - x^2$, an

underbar $-x^4$, $1 : 2'$ –rotation about the axis x^1 through the angle π , and $\cdot m'$ –reflection of x^1 .

for $H_{10}(X_3 + kX_6)$ n , $n : 2$, $\overline{2n}$, $n : m$, $n \cdot m$, $m \cdot n : m$, $\overline{2n} \cdot m$ ($n = 1, 2, \dots, k > 0$);

for $H_{11}(X_6)$ n , $2n$, $n \times 1$, $n : 2$, $n : \underline{2}$, $\underline{2n} : 2$, $n : 2 \times \underline{1}$, $\overline{2n}$, $\underline{2n}$, $\overline{2n} \times 1$, $n : m$, $n : \underline{m}$, $\underline{2n} : m$, $2n : \underline{m}$; $n : m \times \underline{1}$, $n \cdot m$, $n \cdot \underline{m}$, $\underline{2n} \cdot m$, $n \cdot m \times \underline{1}$, $m \cdot n : m$, $\underline{m} \cdot n : m$, $\underline{m} \cdot n : \underline{m}$, $m \cdot n : \underline{m}$, $m \cdot \underline{2n} : m$, $m \cdot \underline{2n} : \underline{m}$, $m \cdot n : m \times 1$, $\overline{2n} \cdot m$, $\overline{2n} \cdot \underline{m}$, $\underline{2n} \cdot m$, $\underline{2n} \cdot m$, $\overline{2n} \cdot m \times 1$ ($n = 1, 2, \dots$).

Thus, taking into account the finite subgroups of the group G , we obtain altogether 207 distinct types of subgroups with finite component group, 86 of which depend on the natural number n . In particular, for $\mu = 9, 12$ one obtains the limiting groups known in crystallography $(\frac{1}{2}, \frac{2}{2})$.

The author expresses his gratitude to L. I. Sedov for a fruitful discussion of the work.

Moscow State University
named after M. V. Lomonosov

Received
11 X 1968

CITED LITERATURE

- ¹ A. V. Shubnikov, *Symmetry and Antisymmetry of Finite Figures*, Publishing House of the Academy of Sciences of the USSR, 1951.
- ² Yu. P. Sirotnin, *Crystallography*, **7**, no. 1, 89 (1962).
- ³ F. Murnaghan, *The Theory of Group Representations*, IL, 1950.
- ⁴ G. I. Kruchkovich, *Proceedings of the Seminar on Vector and Tensor Analysis*, **12**, Moscow, 1963.
- ⁵ V. G. Kosh, *Scientific Notes of Kazan State University*, **123**, book 1, 59 (1963).
- ⁶ C. Chevalley, *Theory of Lie Groups*, **1**, IL, 1948.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.