

PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

N. V. Medvedev

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.79812>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.91+517.948

MATHEMATICS

N. V. MEDVEDEV

PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

(Presented by Academician I. G. Petrovskii on 27 V 1968)

Consider the differential equation

$$dx/dt = f(t, x), \quad (1)$$

where $x = x(t)$ is the unknown function with values in a certain complex Banach space E ; $f(t, x)$ is an operator which, for each fixed t , acts in E and is periodic in t with period 2π . It is assumed that the operator $f(t, x)$ maps a strongly continuous 2π -periodic function $x(t)$ into the likewise strongly continuous function $f(t, x(t))$.

There are various methods for studying periodic solutions of equation (1), for example, ^(1,2). In the present work one more method is proposed for solving the problem of the existence and finding of periodic solutions of equation (1), which is close to Cesari's method ⁽²⁾.

Consider the operator B_1 , defined by the equality

$$B_1 x(t) = a + \frac{1}{2\pi} \int_t^{t+2\pi} (s - t - 2\pi) f(s, x(s)) ds.$$

Let X be the set consisting of strongly continuous 2π -periodic functions. Then the operator B_1 acts in X for any value of the parameter $a \in E$.

Lemma. *Let $x(t, a)$ be a fixed point of the operator B_1 . If*

$$\int_0^{2\pi} f(s, x(s, a)) ds = 0, \quad (2)$$

then the function $x(t, a)$ is a 2π -periodic solution of equation (1).

The operator B_1 is convenient for studying the equation

$$dx/dt = A(t)x + f(t), \quad (3)$$

where $A(t)$ is a linear bounded operator, continuous and periodic in t with period 2π , and $f(t)$ is a continuous 2π -periodic function.

Introduce the following series:

$$U(t) = -\frac{1}{2\pi} \int_{t+2\pi}^t A(s) ds + \frac{1}{2\pi} \int_{t+2\pi}^t \left(\int_{t+2\pi}^s A(s) ds \right) A(s) ds - \dots,$$

$$g(t) = \frac{1}{2\pi} \int_{t+2\pi}^t f(s) ds - \frac{1}{2\pi} \int_{t+2\pi}^t \left(\int_{t+2\pi}^s A(s) ds \right) f(s) ds + \dots$$

In this case we have

$$U(t) = \frac{1}{2\pi} V(t, 0)(V(2\pi, 0) - I)V(0, t),$$

where I is the identity operator, and $V(t, \tau)$ is the solution of the integral equation

$$V(t, \tau) = I + \int_{\tau}^t V(t, s)A(s) ds.$$

Theorem 1. *If equation (3) has a 2π -periodic solution $x(t)$, then the equality $U(t)x(t) + g(t) = 0$ holds.*

The proof is carried out using the operator B_1 .

One can give the following sufficient condition for the invertibility of the operator $U(t)$.

Theorem 2. Suppose

$$\alpha = \|A(0)\|, \quad \gamma = \sup_t \|A(t)\|, \quad \beta = \frac{1}{2\pi} \int_0^{2\pi} \|A(s) - A(0)\| ds.$$

Suppose the operator $I - 2\pi A(0)$ is invertible and

$$\|(I - 2\pi A(0))^{-1}\| = (1 + 2\pi\rho)^{-1}.$$

Then, if

$$\frac{1 + 2\pi\beta \exp(2\pi\gamma)}{1 + 2\pi\rho - 4\pi^2\alpha\gamma \exp(2\pi\gamma)} < 1,$$

equation (3) has one and only one 2π -periodic solution.

We now introduce a more general operator B_2 , where

$$B_2 y(t) = -\varphi(t) + \frac{1}{2\pi} \int_t^{t+2\pi} (s-t-\pi) f(s, a_0 + \varphi(s) + y(s)) ds,$$

$$\varphi(t) = \sum_{0 < |n| \leq k} a_n e^{int}.$$

The operator B_2 , for arbitrary values of the parameters $a_0, a_1, a_{-1}, \dots, a_k, a_{-k}$, also acts in X . If $y(t, a_0, \dots, a_{-k})$ is a fixed point of this operator, then $x(t) = a_0 + \varphi(t) + y(t, a_0, \dots, a_{-k})$ is a 2π -periodic solution of equation (1), when the parameters satisfy equation (2).

Theorem 3. Suppose the operator $f(t, x)$ satisfies the Lipschitz condition

$$\|f(t, x') - f(t, x'')\| \leq p(t) \|x' - x''\|, \quad x', x'' \in E. \quad (4)$$

Suppose $H(t)$, for each fixed t , is a linear bounded operator, continuous and 2π -periodic in t , and satisfying the condition

$$\|H(t)(x' - x'') - f(t, x') + f(t, x'')\| \leq q(t) \|x' - x''\|, \quad x', x'' \in E. \quad (5)$$

Here $p(t)$ and $q(t)$ are certain positive 2π -periodic functions. In addition, suppose that for the operator $\int_0^{2\pi} H(t) dt$ there exists a bounded inverse, and that the inequalities

$$\mu = \sup_t \frac{1}{2\pi} \int_{-\pi}^{\pi} |s| p(s+t+\pi) ds < 1,$$

$$\frac{1}{1-\mu} \left\| \left(\int_0^{2\pi} H(t) dt \right)^{-1} \right\| \int_0^{2\pi} (q(t) + \mu \|H^*(t)\|) dt < 1,$$

hold, where

$$H^*(t) = H(t) - \frac{1}{2\pi} \int_0^{2\pi} H(t) dt.$$

Then equation (1) has one and only one 2π -periodic solution.

Proof. The operator B_2 with one parameter a_0 , i.e., with $\varphi(t) \equiv 0$, is, by (4), a contraction operator. Further, taking into account the identity

$$\begin{aligned} \int_0^{2\pi} H(t) dt a_0 &= \int_0^{2\pi} (H(t)x(t) - f(t, x(t))) dt - \int_0^{2\pi} H^*(t)y(t, a_0) dt + \\ &+ \int_0^{2\pi} f(t, x(t)) dt, \end{aligned}$$

where $x(t) = a_0 + y(t, a_0)$, $y(t, a_0)$ is a fixed point of B_2 , the operator is constructed

$$Fa_0 = \left(\int_0^{2\pi} H(t) dt \right)^{-1} \left[\int_0^{2\pi} (H(t)x(t) - f(t, x(t))) dt - \int_0^{2\pi} H^*(t)y(t, a_0) dt \right].$$

By virtue of (5), the operator F has a fixed point a_0^* . Then the function $x^*(t) = a_0^* + y(t, a_0^*)$ is a periodic solution of equation (1). The theorem is proved.

By a somewhat different method one can obtain a proposition for equation (3).

Theorem 4. *Let the operator $\int_0^{2\pi} A(s) ds$ have a bounded inverse with norm η , and*

$$M = \sup_t \frac{1}{2\pi} \int_{-\pi}^{\pi} \|sA(s+t+\pi)\| ds, \quad N = \int_0^{2\pi} \|A(s)\| ds.$$

Then, if $M(1 + \eta N) < 1$, equation (3) has a 2π -periodic solution.

Proof is carried out by the method of successive approximations, putting

$$\begin{aligned} y_{n+1}(t) &= \frac{1}{2\pi} \int_t^{t+2\pi} (s-t-\pi)[A(s)(a_n + y_n(s)) + f(s)] ds, \\ \int_0^{2\pi} A(s)(a_n + y_n(s)) ds + \int_0^{2\pi} f(s) ds &= 0. \end{aligned}$$

The following proposition is, in a certain sense, a generalization of Theorem 3.

Theorem 5. *Let E be a Banach space such that for every function $x(t) \in X$ the equality*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \|x(t)\|^2 dt = \sum_{n=-\infty}^{\infty} \|a_n\|^2, \quad a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t)e^{-int} dt. \quad (6)$$

holds. Let the operator $f(t, x)$ satisfy condition (4), where $p(t) = p$ is constant. Let there exist a constant linear operator H satisfying condition (5), and suppose that the eigenvalues of the operator H are distinct from numbers of the form ni , $n = 0, \pm 1, \dots, \pm k$, where k is some natural number. Then, if

$$\mu_k = 2p^2 \sum_{n=k+1}^{\infty} \frac{1}{n^2} < 1, \quad (7)$$

$$\frac{1}{2\pi(1 - \mu_k)} \int_0^{2\pi} q^2(t) dt \sum_{|n| \leq k} \|(niI - H)^{-1}\|^2 < 1, \quad (8)$$

then equation (1) has one and only one 2π -periodic solution.

Proof. In the metric space of sequences of the form $\{a_n\}$, $n = \pm(k+1), \pm(k+2), \dots$, $a_n \in E$, $\sum \|a_n\| < +\infty$, an operator Φ is constructed, defined by the right-hand sides of the equalities

$$a_n = \frac{1}{2\pi ni} \int_{-\pi}^{\pi} f(t, a_0 + \varphi(t) + y(t)) e^{-int} dt, \quad |n| > k,$$

where

$$y(t) = \sum_{|n| > k} a_n e^{int}.$$

The operator Φ , by virtue of (7), has a fixed point $\{a_n^*\}$, the coordinates of which depend on the parameters. Next, in the parameter space an operator Ψ is introduced, defined by the right-hand sides of the equalities

$$a_n = \frac{1}{2\pi} (-niI + H)^{-1} \int_{-\pi}^{\pi} (Hx(t) - f(t, x(t))) e^{-int} dt, \quad |n| \leq k,$$

where

$$x(t) = a_0 + \varphi(t) + y^*(t), \quad y^*(t) = \sum_{|n| > k} a_n^* e^{int}.$$

By virtue of (8), the operator Ψ has a fixed point with coordinates a_n^* , $|n| \leq k$. Then, in turn, the function $y^*(t)$, for $a_n = a_n^*$, $|n| \leq k$, is a fixed point of the operator B_2 . The theorem is proved.

We note that for the linear equation (2), when A is a constant matrix, the conditions of Theorem 5 are at the same time necessary. Indeed, in this case it is necessary that the eigenvalues of the matrix A be distinct from numbers

of the form ni , $n = 0, \pm 1, \pm 2, \dots$. Then the role of the operator H is played by the matrix A . In this case condition (8) holds for any k , and condition (7) for sufficiently large k . We also note that Theorem 5 can be formulated using a variable operator $H(t)$.

The preceding method can be generalized by considering equations of higher order. For example, for the equation

$$d^2x/dt^2 = f(t, x), \quad (9)$$

where $f(t, x)$ is the same operator as in equation (1), the following holds.

Theorem 6. *Let the operator $f(t, x)$ satisfy condition (4). Let there exist an operator $H(t)$, satisfying condition (5), such that the eigenvalues of the operator*

$$\int_0^{2\pi} H(t) dt$$

are distinct from numbers of the form $-2\pi n^2$, $n = 0, 1, \dots, k$, where k is some natural number or zero. Then, if

$$\mu_k = \frac{1}{\pi} \int_0^{2\pi} p(t) dt \sum_{n=k+1}^{\infty} \frac{1}{n^2} < 1,$$

$$\frac{1}{1 - \mu_k} \sum_{|n| \leq k} \left\| \left(2\pi n^2 I + \int_0^{2\pi} H(t) dt \right)^{-1} \int_0^{2\pi} (q(t) + \|H^*(t)\|) dt \right\| < 1,$$

then equation (9) has one and only one 2π -periodic solution.

Proof is carried out using the operator

$$B_3 y(t) = -\varphi(t) + \frac{1}{4\pi^2} \int_t^{t+2\pi} (s-t-\pi) \int_s^{s+2\pi} (\tau-s-\pi) f(\tau, a_0 + \varphi(\tau) + y(\tau)) d\tau.$$

We note that if, for any $x(t) \in X$, equality (7) holds, then under the conditions of Theorem 6 the estimates can, in a certain sense, be improved.

Remark. The method presented can without particular difficulty be extended to equations of the form (3)

$$dx/dt = \omega(t, \tilde{x}),$$

where $\tilde{x} = x(t)$, $\omega(t, \tilde{x})$ is an operator with values in E , defined for each t on the set X and periodic in t with period 2π , i.e., the identity $\omega(t + 2\pi, \tilde{x}) = \omega(t, \tilde{x})$ holds for all t and any fixed $\tilde{x} \in X$.

Vladimir Pedagogical Institute

Received
21 V 1968

CITED LITERATURE

1. M. A. Krasnosel' skii, UMN, 21, no. 3 (1966).
2. L. Cesari, Contrib. diff. eq., 1, N. Y., 1963.
3. M. A. Krasnosel' skii, DAN, 152, No. 4 (1963).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.