

GENERALIZED SOLUTIONS OF THE CAUCHY PROBLEM IN THE LARGE FOR NONLINEAR FIRST-ORDER EQUATIONS

S. N. Kruzhkov

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.78698>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.945

MATHEMATICS

S. N. KRUZHKO

GENERALIZED SOLUTIONS OF THE CAUCHY PROBLEM IN THE LARGE FOR NONLINEAR FIRST-ORDER EQUATIONS

(Presented by Academician A. Yu. Ishlinskii, 20 I 1969)

In the present paper we establish nonlocal existence and uniqueness theorems for generalized solutions of the Cauchy problem for quasilinear equations

$$u_t + \sum_{i=1}^n (\varphi_i(u))_{x_i} = 0, \quad \varphi_i(u) \in C^1, \quad (1)$$

and for nonlinear equations

$$u_t + F(u_x) = 0, \quad F(p) \in C^1, \quad n = 1, \quad (2)$$

with the initial condition

$$u|_{t=0} = u_0(x). \quad (3)$$

Various particular cases of problems (1), (3) and (2), (3) (connected with various assumptions on the structure of the equations and on the properties of the initial function $u_0(x)$) have been studied in papers ⁽¹⁻¹³⁾, etc.

1. The Cauchy problem for equation (1) in the class of bounded measurable functions

Let $u_0(x)$ be an arbitrary bounded measurable function in n -dimensional Euclidean space E_n .

Definition 1. A bounded measurable function $u(t, x)$ is called a **generalized solution** of problem (1), (3) in the layer $\Pi_T = [0, T] \times E_n$ if: 1) for any constant k and any smooth finite function $f(t, x) \geq 0$ in Π_T , the inequality

$$\iint_{\Pi_T} \left\{ |u - k| f_t + \sum_{i=1}^n \operatorname{sign}(u - k) [\varphi_i(u) - \varphi_i(k)] f_{x_i} \right\} dx dt \geq 0; \quad (4)$$

holds; 2) there exists a set $\mathcal{E}$ of zero measure on $[0, T]$ such that for $t \in [0, T] \setminus \mathcal{E}$ the function $u(t, x)$ is defined almost everywhere in E_n , and for every ball $K_R = \{|x| \leq R\} \subset E_n$

$$\lim_{\substack{t \rightarrow 0 \\ t \in [0, T] \setminus \mathcal{E}}} \int_{K_R} |u(t, x) - u_0(x)| dx = 0. \quad (5)$$

Obviously, from inequality (4), for $k = \pm \sup |u(t, x)|$, by virtue of the arbitrariness of the function $f \geq 0$, it follows that the function $u(t, x)$ satisfies equation (1) in the sense of the integral identity

$$\iint_{\Pi_T} \left[u f_t + \sum_{i=1}^n \varphi_i(u) f_{x_i} \right] dx dt = 0,$$

valid for any smooth and finite function f in Π_T .

It is essential that Definition 1 also contains a condition characterizing admissible discontinuities of solutions; if in some neighborhood of a discontinuity point the generalized solution is piecewise smooth, then, by integration by parts in (4), taking into account arbitrariness in the choice of f , this condition is easily localized and written in the form of a pointwise inequality along the discontinuity surface, which in the case $n = 1$ coincides with the condition E ,

formulated in the work ⁽⁶⁾ (see also ⁽⁵⁾), and which is a natural analogue of the gas-dynamical condition of entropy increase at discontinuities (see ⁽¹⁴⁾, § 82).

Theorem 1. *A generalized solution of problem (1), (3) exists.**

To prove Theorem 1, the method of “vanishing viscosity” is applied. The Cauchy problem is considered for the parabolic equation

$$u_t + \sum_{i=1}^n (\varphi_i(u))_{x_i} = \varepsilon \Delta u, \quad \varepsilon \in (0, 1), \quad (6)$$

with initial condition (3), and for solutions $u^\varepsilon(t, x)$ of problem (6), (3) the estimate is established

$$\int_{K_R} |u^\varepsilon(t + \tau, x) - u^\varepsilon(t, x + h)| dx \leq \omega(\tau) + \omega(|h|). \quad (7)$$

Here $t \in [0, T]$, $\tau \geq 0$, and the continuous monotone function $\omega(\sigma)$ ($\omega(0) = 0$) does not depend on ε (see ⁽¹⁵⁾). It is easy to see that the solution $u^\varepsilon(t, x)$ satisfies the inequality

$$\iint_{\Pi_T} \left[\psi(u)(f_t + \varepsilon \Delta f) + \sum_{i=1}^n \int_k^u \psi'(u) \varphi'_i(u) du f_{x_i} \right] dx dt \geq 0 \quad (8)$$

for any twice smooth function $f(t, x) \geq 0$ and any convex downward function $\psi(u)$. Letting ε tend to zero in (8) with $\psi = |u - k|$, taking into account estimate (7) (which ensures compactness of the family $\{u^\varepsilon\}$ in L_1), we obtain the existence of a solution of problem (1), (3).

Theorem 2. *Let the functions $u(t, x)$ and $v(t, x)$ be solutions of problem (1), (3) with initial functions $u_0(x)$ and $v_0(x)$, respectively, and let $|u(t, x)| \leq M$, $|v(t, x)| \leq M$; let S_t be the section of the cone $\{(t, x) : |x| \leq R + N(T - t), 0 \leq t \leq T\}$ by the plane $t = \tau = \text{const}$, where*

$$N = \max \left(\sum_{i=1}^n \varphi_i^2(u) \right)^{1/2} \quad \text{for } |u| \leq M.$$

Then for almost all $\tau \in [0, T]$ the estimate holds

$$\int_{S_\tau} |u(\tau, x) - v(\tau, x)| dx \leq \int_{S_0} |u_0(x) - v_0(x)| dx. \quad (9)$$

Corollary 1. *The generalized solution of problem (1), (3) in the sense of Definition 1 is unique.*

Corollary 2. *Let a family $\{u^\nu(t, x)\}$ of generalized solutions of problem (1), (3) be given, with $|u^\nu(t, x)| \leq M = \text{const}$, and suppose the initial functions $u_0^\nu(x)$ are relatively continuous in the L_1 norm in every ball K_R . Then the family $\{u^\nu\}$ is compact in the L_1 norm on every bounded set in Π_T .*

Corollary 3. *Taking into account the method of constructing generalized solutions of problem (1), (3) indicated in Theorem 1, it is not difficult to derive that if for almost all $x \in E_n$*

$$u(0, x) \geq v(0, x),$$

then for almost all $(t, x) \in \Pi_T$

$$u(t, x) \geq v(t, x).$$

We give here a **brief exposition of the proof of Theorem 2**. Let a smooth function $g(t, x, \tau, y) \geq 0$ be finite in $\Pi_T \times \Pi_T$; in inequality (4) for the function $u(t, x)$ put $k = v(\tau, y)$, $f(t, x) = g$ for fixed (τ, y) , and integrate over Π_T (with respect to τ and y); then in inequality (4) for the function $v(\tau, y)$ put $k = u(t, x)$, $f(\tau, y) = g$ for fixed (t, x) , and integrate over Π_T (with respect to t and x). Adding the two inequalities obtained for the integrals over $\Pi_T \times \Pi_T$, we find,

* This result was formulated by me at one of the meetings of the seventh section of the International Congress of Mathematicians in Moscow in 1966.

that

$$\begin{aligned} & \iint_{\Pi_T} \iint_{\Pi_T} \{ |u(t, x) - v(\tau, y)| (g_t + g_\tau) + \\ & + \sum_{i=1}^n \text{sign}(u(t, x) - v(\tau, y)) [\varphi_i(u(t, x)) - \\ & - \varphi_i(v(\tau, y))] (g_{x_i} + g_{y_i}) \} dx dt dy d\tau \geq 0. \end{aligned} \quad (10)$$

Let now $f(t, x)$ be an arbitrary test function from Definition 1, and let $\{\delta_\nu(\sigma) \geq 0\}$ be a delta-like sequence of smooth functions on $[-T, T]$, $\delta_\nu(\sigma) \rightarrow \delta(\sigma)$ as $\nu \rightarrow \infty$, where $\delta(\sigma)$ is concentrated at the point $\sigma = 0$ and $\delta_\nu(\sigma) \equiv 0$ for $|\sigma| \geq h$, with the number h such that $f(t, x) \equiv 0$ for $t \in [0, T] \setminus [h, T - h]$. Put in (10)

$$g = f\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \delta_\nu\left(\frac{t-\tau}{2}\right) \prod_{i=1}^n \delta_\nu\left(\frac{x_i - y_i}{2}\right) \equiv f(\cdots) \lambda_\nu,$$

noting that $g_t + g_\tau = f_t(\cdots) \lambda_\nu$, $g_{x_i} + g_{y_i} = f_{x_i}(\cdots) \lambda_\nu$, $(\cdots) = (\frac{t+\tau}{2}, \frac{x+y}{2})$; letting $\nu \rightarrow \infty$, we obtain

$$\begin{aligned} & \iint_{\Pi_T} \{ |u(t, x) - v(t, x)| f_t + \\ & + \sum_{i=1}^n \text{sign}(u(t, x) - v(t, x)) [\varphi_i(u(t, x)) - \varphi_i(v(t, x))] f_{x_i} \} dx dt \geq 0. \end{aligned} \quad (11)$$

Define

$$\alpha_\nu(\sigma) = \int_{-\infty}^{\sigma} \delta_\nu(\sigma) d\sigma, \quad \delta_\nu(\sigma) \equiv 0 \quad \text{for } |\sigma| \leq \nu^{-1}.$$

Let the numbers t_0 and $\tau \in (0, T)$, $t_0 < \tau$, and $\nu \geq \max[t_0^{-1}, (T - \tau)^{-1}]$. Put in (11) $f = [\alpha_\nu(t - t_0) - \alpha_\nu(t - \tau)]\chi(t, x)$, where $\chi(t, x) = \chi_{\nu_0} \equiv 1 - \alpha_{\nu_0}(|x| - N(T - t) - R)$, and note that for the function $\chi(t, x)$ the relations hold: $0 = \chi_t + N|\chi_x| \geq \chi_t + \sum_{i=1}^n [(\varphi_i(u) - \varphi_i(v)) \times$

$$\times (u - v)^{-1} \chi_{x_i}.$$

Letting in (11) (with the indicated choice of f) successively ν and (then) ν_0 tend to infinity, we obtain that for almost all t_0 and $\tau \in [0, T]$ the inequality

$$\int_{S_\tau} |u(\tau, x) - v(\tau, x)| dx \leq \int_{S_{t_0}} |u(t_0, x) - v(t_0, x)| dx,$$

holds, whence, in the limit as $t_0 \rightarrow 0$ (along the corresponding sequence), estimate (9) follows.

Remark. By virtue of the uniqueness of the generalized solution $u(t, x)$ of problem (1), (3), the sequence $\{u^\varepsilon(t, x)\}$ of solutions of problem (6), (3) converges to $u(t, x)$ in the L_1 norm on any compact set in Π_T as ε tends arbitrarily to zero; this justifies the introduction of the vanishing viscosity in equation (1).

2. The Cauchy problem for equation (2) in the class of functions continuous in the Lipschitz sense. Let the function $u_0(x)$ satisfy the Lipschitz condition on the line E_1 .

Definition 2. A Lipschitz-continuous function $u(t, x)$ is called a **generalized solution of problem (2), (3)** in the strip $\Pi_T =$

$= [0, T] \times E_1$, if: 1) the function $u(t, x)$ satisfies equation (2) almost everywhere in Π_T and assumes the initial values (3); 2) for any constant k and any smooth finite function $f(t, x) \geq 0$ in Π_T , the inequality

$$\iint_{\Pi_T} \{ |u_x - k| f_t + \text{sign}(u_x - k) [F(u_x) - F(k)] f_x \} dx dt \geq 0$$

holds, and the derivative $u_x(t, x)$ is continuous in x in L_1 on any segment uniformly with respect to $t \in [0, T]$.

Theorem 3. *Problem (2), (3) has a unique generalized solution in the sense of Definition 2. This solution can be obtained as the limit as $\varepsilon \rightarrow +0$ of the solutions $u^\varepsilon(t, x)$ of the Cauchy problem for the parabolic equation*

$$u_t + F(u_x) = \varepsilon u_{xx}, \quad \varepsilon \in (0, 1) \quad (12)$$

with the initial condition (3) (the convergence is uniform on any compact set).

Theorem 4. *Let the functions $u(t, x)$ and $v(t, x)$ be generalized solutions of problem (2), (3) with initial functions $u_0(x)$ and $v_0(x)$, respectively. Then for $t \in [0, T]$ the estimate*

$$\sup_x |u(t, x) - v(t, x)| \leq \sup_x |u_0(x) - v_0(x)|$$

holds.

Theorem 4 admits a natural refinement connected with the introduction of the notion of a domain of dependence.

Moscow State University
named after M. V. Lomonosov

Received
13 I 1969

REFERENCES

1. E. Hopf, Comm. Pure and Appl. Math., 3, No. 3, 201 (1950).
2. O. A. Oleinik, DAN, 95, No. 3, 451 (1954).
3. A. N. Tikhonov, A. A. Samarskii, DAN, 99, No. 1, 27 (1954).
4. P. Lax, Comm. Pure and Appl. Math., 7, No. 1, 159 (1954).
5. I. M. Gelfand, UMN, 14, No. 2, 87 (1959).
6. O. A. Oleinik, UMN, 14, No. 2, 165 (1959).
7. A. Douglis, Comm. Pure and Appl. Math., 12, No. 1, 87 (1959).
8. S. N. Kruzhkov, DAN, 132, No. 1, 36 (1960).
9. A. Douglis, Comm. Pure and Appl. Math., 14, No. 3, 267 (1961).
10. S. N. Kruzhkov, UMN, 20, No. 6, 112 (1965).
11. E. Conway, J. Smoller, Comm. Pure and Appl. Math., 19, No. 1, 95 (1966).
12. A. I. Volpert, Matem. sborn., 73, No. 2, 255 (1967).
13. N. N. Kuznetsov, Matem. zametki, 2, No. 4, 401 (1967).

14. L. D. Landau, E. M. Lifshitz, *Fluid Mechanics*, Moscow, 1953.

15. S. N. Kruzhkov, Matem. zametki, 6, No. 1 (1969).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.