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Abstract

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MATHEMATICS

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CATEGORIES OF CORRESPONDENCES OVER SEMI-ABELIAN CATEGORIES

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An axiomatic description is given of the categories of correspondences over semi-abelian categories considered in ⁽⁴⁾. In doing so, first the case of categories without direct sums and products— β -categories—is studied, and then the semi-abelian categories themselves.

For the notions of category theory and the properties of various categories used in this paper without references, see ^(1, 4, 5).

1. We shall call a category $\mathcal{K}$ a β -category if it satisfies the following axioms:

SA1(A3). In the category $\mathcal{K}$ there exists a zero object O .

SA2. For any morphism α of $\mathcal{K}$ there exists a cokernel $\text{Coker } \alpha$.

A4.a) For a diagram

$$\begin{array}{c} \alpha \\ \rightarrow \uparrow \mu, \end{array}$$

where μ is a monomorphism, there exists a couniversal square

$$\begin{array}{ccc} & \alpha & \\ \mu' \uparrow & \dashrightarrow & \uparrow \mu \\ & \alpha' & \end{array} \quad (1)$$

b) moreover, if in the couniversal square (1) α is a normal epimorphism, then α' is also a normal epimorphism.

From axioms SA1 and A4 it follows that:

- 1) for any morphism α of $\mathcal{K}$ there exists a kernel $\text{Ker } \alpha$;
- 2) for any two monomorphisms $\mu_i : B_i \rightarrow A$, $i = 1, 2$, there exists their intersection $\mu = \mu_1 \cap \mu_2 : B \rightarrow A$.

SA3. For any morphism α of $\mathcal{K}$ there exists a canonical decomposition

$$\alpha = (\text{Coker } \text{Ker } \alpha) \dot{\alpha} (\text{Ker } \text{Coker } \alpha),$$

in which α is a bimorphism.

A1. $\mathcal{K}$ is a locally small category on the left, i.e. the subobjects $\bigcup(A)$ of an object $A \in \mathcal{K}$ form a set.

A7(SA4). If in the diagram (2) the rows are exact in the sense of ⁽⁴⁾, then α is an equivalence

$$\begin{array}{ccccccc} 0 & \rightarrow & \cdot & \xrightarrow{\mu} & \cdot & \xrightarrow{\pi} & \cdot \rightarrow 0 \\ & & 1 \uparrow & & \uparrow \alpha & & \uparrow 1 \\ 0 & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot \rightarrow 0 \\ & & \rho & & \nu & & \end{array} \quad (2)$$

It is easy to see that, for β -categories, properties 1–6 from ⁽⁴⁾ are valid. In addition, in β -categories axiom SA6 always holds, as do such consequences of it as 12–15 (see ⁽⁴⁾), and also the properties:

I. Any pair of morphisms φ_1, φ_2 from $\mathcal{K}$ has a left equalizer.

II. The left equalizer is a normal monomorphism.

For the proof of these properties, axioms A4 and SA3 are used.

Let R be a category with involution (see ⁽¹⁾), satisfying the axioms:

K1. In R there exists an I -zero object O .

K2. a) If $If \subset Ig$, then $gf^*f \subset g$.

b) If $Bf \supset Bg$, then $gf^*f \supset g$.

K3. For every morphism $u \in R(0, A)$ there exists an injection $m \in R(U, A)$ for which $Bm = u$.

Various properties of such categories were obtained in the papers ^(1,3). In particular, the subcategory $G(R)$ of proper morphisms of the category R satisfies axioms A1, A3, A4, A7 (see ⁽³⁾), and for any morphisms $u, v \in R(0, A)$ there exists an intersection $u \cap v \in R(0, A)$.

A morphism $u \in R(0, A)$ will be called **strict** if, for every D -regular morphism $f \in R(A, B)$ and every $v \in R(0, A)$,

$$((u \subset Kf \Rightarrow v \subset Kf) \Rightarrow v \subset u),$$

where $R(A, B)$ is the set of morphisms from an object $A \in R$ to an object $B \in R$. We note that a morphism equal to Kf for some D -regular morphism f is always strict.

K7⁰. For every strict morphism $u \in R(0, A)$ there exists a projection $p \in R(A, U)$ for which $Kp = u$.

K8. For every morphism $v \in R(0, A)$ there exists a minimal strict morphism $u \in R(0, A)$ for which $u \supset v$.

K9. If an injection m is a left equalizer in the category $G(R)$, then Bm is a strict morphism.

In a β -category $\mathcal{K}$ one can define a correspondence between objects $A, B \in \mathcal{K}$ by means of a construction analogous to the construction of correspondences in quasi-exact and γ -categories (see ^(2,3)), and show, as in the paper ⁽³⁾, that the following is true.

Theorem 1. For an arbitrary β -category $\mathcal{K}$ there exists an extension to a category with involution $R(\mathcal{K})$, satisfying axioms K1–K3, K7⁰–K9, and the category $\mathcal{K}$ is isomorphic to the subcategory $G(R(\mathcal{K}))$ of proper morphisms of the category $R(\mathcal{K})$.

Let R be a category with involution satisfying axioms K1, K2.

Lemma. If m, n are injections, and Bm, Bn are strict morphisms, then Bmn is a strict morphism ($m \in R(A, B)$, $n \in R(B, C)$).

With the help of this lemma and axioms K7⁰–K9 it is verified that the category $G(R)$ satisfies axioms SA2 and SA3, i.e.

Theorem 2. If a category with involution R satisfies axioms K1–K3, K7⁰–K9, then its subcategory of proper morphisms $G(R)$ is a β -category.

Theorem 3. Between β -categories and categories with involution satisfying axioms K1–K3, K7⁰–K9, there exists a one-to-one correspondence up to equivalence of categories.

The proof is carried out in the same way as for γ -categories (see ⁽³⁾), relying on Theorems 1, 2 and the following

Proposition. If categories with involution R and R' satisfy axioms K1–K3, K7⁰–K9, then:

- 1) an I -functor (see ⁽¹⁾) $Q : R \rightarrow R'$ induces an exact functor

$$BQ : G(R) \rightarrow G(R'),$$

preserving intersections of subobjects;

- 2) every exact functor $F : G(R) \rightarrow G(R')$, preserving intersections of subobjects, can be uniquely extended to an I -functor

$$EF : R \rightarrow R'.$$

2. Let the β -category $\mathcal{K}$ satisfy axiom A8. For any objects $A, B \in \mathcal{K}$ there exists a direct product

$$A \times B \in \mathcal{K}.$$

Then, just as in the case of abelian categories (see ⁽¹⁾), one can show that the β -category $\mathcal{K}$ will be additive (in the sense of Grothendieck ⁽⁶⁾), i.e. it satisfies axiom SA5 and is semiabelian (see ⁽⁴⁾). In this case, instead of axiom A8 one

may require the existence in the category $\mathcal{K}$ of a couniversal square for an arbitrary diagram

or the fulfillment of the axiom dual to A8:

A8'. For any objects $A, B \in \mathcal{K}$ there exists a direct sum (free product)

$$A \oplus B \in \mathcal{K}.$$

Let a category with involution R satisfy axiom K1. We introduce some additional conditions (see (1)):

K4. In the category R , for any pair of morphisms $f, g \in R(A, B)$, $A, B \in R$, there exist:

- a) their intersection $f \cap g \in R(A, B)$;
- b) their union $f \cup g \in R(A, B)$.

Remark. The intersection of strict morphisms (see § 1) is a strict morphism.

K5. Let $f \in R(A, B)$, $g_1, g_2 \in R(B, C)$. Then:

- a) from $If \subset Kg_1$ it follows that

$$f(g_1 \cap g_2) \supset fg_1 \cap fg_2;$$

- b) from $Bf \supset Dg_1$ it follows that

$$f(g_1 \cup g_2) \subset fg_1 \cup fg_2.$$

We note that the relations

$$f(g_1 \cap g_2) \subset fg_1 \cap fg_2$$

and

$$f(g_1 \cup g_2) \subset fg_1 \cup fg_2$$

follow from the monotonicity of multiplication and from the properties of intersection and union of morphisms.

K6. For any objects $A_1, A_2 \in R$ there exist:

- a) an object $P \in R$ and proper morphisms $g_i : P \rightarrow A_i$, $i = 1, 2$, for which

$$g_1^* g_2 = \Omega_{A_1 A_2};$$

- b) an object $S \in R$ and proper morphisms $h_i : A_i \rightarrow S$, $i = 1, 2$, for which

$$h_1 h_2^* = \omega_{A_1 A_2}.$$

Theorem 4. 1) In a category with involution R satisfying axioms K1–K3, K7⁰–K9, the conditions K4a)–K6a) and K4b)–K6b) are equivalent. 2) If a category with involution R satisfies axioms K1–K9, then its subcategory $G(R)$ is semi-abelian.

The proof is carried out analogously to that of D. Puppe (¹) for the abelian case, relying here on the results of the preceding paragraph.

In a semi-abelian category $\mathcal{K}$ one can, in the usual way, introduce correspondences as subobjects of direct products and consider the category of correspondences $R(\mathcal{K})$ (see (⁴)).

Theorem 5. 1) The category of correspondences $R(\mathcal{K})$ over a semi-abelian category $\mathcal{K}$ satisfies axioms K1–K9. 2) Between semi-abelian categories and categories with involution satisfying axioms K1–K9 there is a one-to-one correspondence up to equivalence of categories.

In (⁴) it is shown that the category $R(\mathcal{K})$ satisfies axioms K1–K6 and that the category $\mathcal{K}$ is isomorphic to the subcategory $G(R(\mathcal{K}))$ of proper morphisms of the category $R(\mathcal{K})$. Verifying axioms K7⁰–K9, we obtain the first assertion of the theorem, and, using the assertion from § 1, also the second.

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