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Abstract

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MATHEMATICS

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MIXED CONTROL IN A DIFFERENTIAL GAME

Let us consider a controlled system described by the equation

$$\dot{x} = f^{(1)}(t, x, u) - f^{(2)}(t, x, v), \quad (1)$$

where x is an n -dimensional phase vector; u and v are r -dimensional control vectors belonging to the first and second players, respectively; the realizations $u[t]$ and $v[t]$ are constrained by

$$u[t] \in \mathcal{U}, \quad v[t] \in \mathcal{V}, \quad (2)$$

where $\mathcal{U}$ and $\mathcal{V}$ are bounded and closed sets; the functions $f^{(i)}$ are continuous in all arguments, differentiable with respect to x , and satisfy the inequalities

$$|x' f_x^{(i)}(t, x, w)| \leq \lambda (1 + \|x\|^2) \quad (\lambda = \text{const}; i = 1, 2).$$

(Vectors are treated as column vectors, the symbol $\|q\|$ denotes the Euclidean norm of the vector q , and a prime as a superscript denotes transposition.)

The problem of the first player is to bring the phase point $x[t]$ to a prescribed set $\mathcal{M}$; the second player prevents this. Similar game situations have been considered, for example, in the works ⁽¹⁻⁷⁾. In the present note we indicate conditions for the successful termination of the game by the first player when his strategy U^0 is formalized within the framework of the theory of differential equations in contingencies ^(8,9). This strategy, which mixes the controls u , is based on the rule of extremal aiming generalized here ⁽⁷⁾, formalized earlier for the differential game of pursuit in ^(10,11). The strategy U^0 proves to be effective in cases of stable absorption ⁽¹²⁾, which correspond to solutions of game control problems under conditions of discrimination of the second player ⁽²⁻⁶⁾.

Admissible strategies U and V are defined as follows. Denote by the symbol $F^{(i)}(t, x)$ the convex hull of the set $\{f = f^{(i)}(t, x, w) : w = u \in \mathcal{U} \text{ or } w = v \in \mathcal{V}\}$. A strategy $W = U$ ($W = V$) will be specified by a system of vector sets $F^{(1)}(t, x; U)$ ($F^{(2)}(t, x; V)$), assigned to each possible position $\{t, x\}$; it will be

admissible (on the interval $t_0 \leq t \leq \vartheta$) if the inclusions $F^{(i)}(t, x; W) \in F^{(i)}(t, x)$ hold and if the sets $F^{(i)}(t, x; W)$ are convex and upper semicontinuous with respect to inclusion relative to variation of t and x for all x and for almost all $t \in [t_0, \vartheta]$. The initial position $\{t_0, x_0\}$ and a pair of admissible strategies $\{U, V\}$ generate motions $x[t]$ ($t_0 \leq t \leq \vartheta$) of system (1), which are absolutely continuous functions satisfying, for almost all t , the equation $\dot{x}[t] = f^{(1)}[t] - f^{(2)}[t]$, where

$$f^{(1)}[t] \in F^{(1)}(t, x[t]; U), \quad f^{(2)}[t] \in F^{(2)}(t, x[t]; V).$$

Let $\mathcal{M}_0$ be some bounded, convex, and closed set contained in $\mathcal{M}$. We shall say that ϑ is the moment of absorption of the set $\mathcal{M}_0$ by the process $x(t)$ (1) from the position $\{t_*, x_*\}$ ($t_* \leq \vartheta$), if the following condition is fulfilled:

1°. To each measurable function $v(t) \in \mathcal{V}(t_* \leq t \leq \vartheta)$ and to any number $a > 0$ one can assign a measurable function $u(t) \in \mathcal{U}(t_* \leq t \leq \vartheta)$ such that the pair of controls $\{u(t), v(t)\}$ transfers the motion $x(t)$ (1) from the position $\{t_*, x_*\}$ to the state $x(\vartheta)$, whose distance from the set $\mathcal{M}_0$ does not exceed a .

Let the symbol $\mathcal{W}(t_*, \vartheta)$ ($t_* \leq \vartheta$) denote the set of all those x for which ϑ is an absorption time of the set $\mathcal{M}_0$ by the process $x(t)$ (1) from the position $\{t_*, x\}$. We shall say that, for $t_0 \leq t \leq \vartheta$, absorption is stable if the following conditions are satisfied.

2°. The sets $\mathcal{W}(t, \vartheta)$ are convex.

3°. Whatever $t_* < \vartheta$, if $x_* \in \mathcal{W}(t_*, \vartheta)$, then for each positive number $\Delta \leq \Delta_0$, $\Delta \leq \vartheta - t_*$ ($\Delta_0 > 0$ —const), for every pair $\{v(t), a\}$ ($v(t) \in \mathcal{V}$, $t_* \leq t \leq t_* + \Delta$, $a > 0$) one can specify a function $u(t) \in \mathcal{U}(t_* \leq t \leq t_* + \Delta)$ such that the pair of controls $\{u(t), v(t)\}$ transfers the motion $x(t)$ (1) from the position $\{t_*, x_*\}$ to the state $x(t_* + \Delta)$, whose distance from the set $\mathcal{W}(t_* + \Delta, \vartheta)$ does not exceed a .

Thus, let some set $\mathcal{M}_0 \subset \mathcal{M}$ be chosen and let the number ϑ be fixed. Construct the strategy U^0 . If the position $\{t, x\}$ ($t < \vartheta$) is such that $x \in \mathcal{W}(t, \vartheta)$, then we set $F^{(1)}(t, x; U^0) = F^{(1)}(t, x)$. If, however, $x \notin \mathcal{W}(t, \vartheta)$, then we construct the smallest closed ε -neighborhood $\mathcal{W}_\varepsilon^0(t, x)(t, \vartheta)$ of the set $\mathcal{W}(t, \vartheta)$ containing the point x . Let s^0 be a unit vector of the inward normal to the boundary of the domain $\mathcal{W}_\varepsilon^0$ at the point x . Then, as $F^{(1)}(t, x; U^0)$, we choose the set of all vectors $f^0 \in F^{(1)}(t, x)$ satisfying the maximum conditions

$$s^{0'} f^0 = \max_f s^{0'} f \quad \text{for } f \in F^{(1)}(t, x). \quad (3)$$

Theorem. Let $\vartheta > t_0$ be an absorption time for the initial position $\{t_0, x_0\}$. If, on the interval $t_0 \leq t \leq \vartheta$, the process $x(t)$ (1) absorbs the set $\mathcal{M}_0$ stably, then the admissible strategy U^0 ensures that the point $x[t]$ is brought onto the set $\mathcal{M}_0$ no later than at the time $t = \vartheta$, whatever the admissible strategy V and whatever motion $x[t]$ possible under the chosen strategies U^0 and V may be.

The main point of the proof is the verification of the estimate $\varepsilon^0[t] \leq \varepsilon^0[t_*] \exp \chi(t - t_*)$ ($\varepsilon^0[t] = \varepsilon^0(t, x[t])$, $\chi = \text{const}$, $t \geq t_*$), which is valid under the conditions of stable absorption along any motion $x[t]$ generated by the strategies U^0 and V . Let us also note that continuous dependence of the sets $\mathcal{W}(t, \vartheta)$ on t is not assumed; however, in our case, as t increases, jump-like changes of $\mathcal{W}(t, \vartheta)$ in the direction of contraction are impossible.

An effective formulation of the absorption conditions, the construction of the sets $\mathcal{W}(t, \vartheta)$ needed for determining the strategy U^0 (3), and the verification of stability of absorption in the general case are difficult. If equation (1) is linear, i.e.,

$$\dot{x} = A(t)x + B(t)u - C(t)v + f(t), \quad (4)$$

then the set $\mathcal{W}(t, \vartheta)$ can be defined as the totality of all those points x that satisfy the inequality

$$\begin{aligned} & \rho^{(1)}(t, \vartheta, l) + \rho^0(l) - \rho^{(2)}(t, \vartheta, l) + \\ & + l' X(\vartheta, t)x + l' \int_t^{\vartheta} X(\vartheta, \tau) f(\tau) d\tau \geq 0, \end{aligned} \quad (5)$$

whatever the vector l may be. Here $X(t, \tau)$ is the fundamental matrix of solutions of the homogeneous part of equation (4), and the quantities $\rho^{(i)}$ and ρ^0 are defined

by the equalities

$$\begin{aligned} \rho^{(1)}(t, \vartheta, l) &= \max_{u(\tau) \in \mathcal{U}^*} \left(\int_t^{\vartheta} l' X(\vartheta, \tau) B(\tau) u(\tau) d\tau \right), \\ \rho^{(2)}(t, \vartheta, l) &= \max_{v(\tau) \in \mathcal{V}^*} \left(\int_t^{\vartheta} l' X(\vartheta, \tau) C(\tau) v(\tau) d\tau \right), \\ \rho^0(l) &= \max_q l' q \quad \text{for } -q \in \mathcal{M}_0, \end{aligned}$$

where $\mathcal{U}^*$ and $\mathcal{V}^*$ are the convex hulls of the sets $\mathcal{U}$ and $\mathcal{V}$ from (2). Condition (5) is equivalent to the absorption conditions in the case of a linear differential pursuit game, where the instant ϑ is determined from the condition of absorption of the attainability domain of the pursued object by the attainability domain of the pursuer (^{7, 10, 11}). Namely, in the present case, with the original problem one should associate the problem of pursuing one controlled object

$$\dot{z} = A(t)z + C(t)v$$

by another

$$\dot{y} = A(t)y + B(t)u + f(t),$$

where the goal of pursuit must be the capture of the point $z[t]$ in a “ $\mathcal{P}$ -neighborhood” of the point $y[t]$, according to the condition

$$z[t] - y[t] = p \in \mathcal{P} = -\mathcal{M}_0.$$

For the position $\{t_*, y_*, z_*\}$ ($y_* - z_* = x_*$), the instant ϑ is then determined by the inclusion

$$G^{(2)}(t_*, z_*, \vartheta) \subset G_P^{(1)}(t_*, y_*, \vartheta),$$

where $G^{(2)}$ is the attainability domain by the instant ϑ for the motion $z(t)$ from the state $z(t_*) = z_*$ under controls $v(t) \in \mathcal{V}^*$, and $G_P^{(1)}$ is the attainability domain by the instant ϑ for the motion $y(t)$ from the state $y(t_*) = y_*$ under controls

$$u_p(t) = u(t) + p\delta(t - \vartheta) \quad (u(t) \in \mathcal{U}^*, \quad p \in \mathcal{P}, \quad \delta(t) \text{ is the delta function}).$$

The stability of absorption is then guaranteed in any case if the boundary H_{P,ε^0} of the minimal ε -neighborhood $G_{P,\varepsilon^0}^{(1)}$ of the domain $G_P^{(1)}$, containing the domain $G^{(2)}$, for every $\varepsilon^0 > 0$ intersects this domain $G^{(2)}$ in the set Q^0 , which is contained entirely in one hyperplane. Then condition (3) is equivalent to the condition of extremal aiming ^(7, 10). Analogous conditions for the stability of absorption are also formulated for the rough nonlinear cases (1), similar to those considered in ⁽¹¹⁾, but they are probably not very effective. The described method of assigning the strategy U^0 by contingencies (3) can also be based on those sets $\mathcal{W}(t, \vartheta)$ which are given by the constructions described in ⁽²⁻⁶⁾, where the control $u[t]$ is implemented with allowance for $v[t]$. The formalization of motions within the framework of differential equations in contingencies then makes it possible to form the control $u[t]$ without resorting to the indicated discrimination of the second player. Finally, we note that in a substantive sense the formal strategy U^0 is revealed in discrete approximating schemes similar to those described in articles ⁽¹⁰⁻¹²⁾.

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