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Abstract

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MATHEMATICS

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ON ONE BOUNDARY-VALUE PROBLEM FOR A MODEL EQUATION OF MIXED TYPE WITH TWO PERPENDICULAR LINES OF DEGENERATION

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In the finite simply connected domain Ω , bounded by the arcs $AB : x^2 + y^2 = 1, x \geq 0, y \geq 0$; $B^*A^* : x^2 + y^2 = 1, x \leq 0, y \leq 0$ and the segments $BB^* : y - x = 1, y \geq 0$; $A^*A : x - y = 1, y \leq 0$, consider the equation of mixed type

$$u_{xx} + \operatorname{sgn}(xy)u_{yy} = 0. \quad (1)$$

Let $CO(OD)$ be the segment $-\frac{1}{2} \leq x \leq 0$ ($0 \leq x \leq \frac{1}{2}$) of the characteristic $x + y = 0$ of equation (1); $BA^*(AB^*)$ the segment of the straight line $x = 0$ ($y = 0$); $\Omega_1(\Omega_1^*)$ and $\Omega_2(\Omega_2^*)$ the hyperbolic parts of the mixed domain Ω , where $x > 0, x + y > 0$ ($x > 0, x + y < 0$) and $x < 0, x + y > 0$ ($x < 0, x + y < 0$), respectively; $\Omega_3(\Omega_3^*)$ the elliptic part of Ω , where $x > 0, y > 0$ ($x < 0, y < 0$);

$$\Delta = \bigcup_{i=1}^3 \Omega_i \cup OA \cup OB, \quad \Delta^* = \bigcup_{i=1}^3 \Omega_i^* \cup OA^* \cup OB^*.$$

Problem A. It is required to determine a function $u(x, y)$ with the following properties: 1) $u(x, y) \in C(\bar{\Omega}) \cap C^{(1,0)}(\Omega \setminus CD)$; 2) $u_x(x, 0), u_x(0, y), u_y(x, 0), u_y(0, y)$ may tend to ∞ of order less than unity at the points A, B, A^*, B^* and O ; 3) $u(x, y)$ is a regular solution in $\Omega \setminus (BA^* \cup AB^* \cup CD)$ of equation (1), satisfying the boundary conditions

$$u|_{AB} = \varphi_1(\theta), \quad 0 \leq \theta \leq \pi/2; \quad u|_{B^*A^*} = \varphi_2(\theta), \quad \pi \leq \theta \leq 3\pi/2; \quad (2)$$

$$u|_{BC} = \psi_1(y), \quad \frac{1}{2} \leq y \leq 1; \quad u|_{A^*D} = \psi_2(y), \quad -1 \leq y \leq -\frac{1}{2}.$$

With respect to the prescribed functions it is assumed that

$$\begin{aligned}\varphi_i(\theta) &= \sin^2 2\theta \bar{\varphi}_i(\theta), \quad i = 1, 2; \quad \bar{\varphi}_1(\theta) \in C(0 \leq \theta \leq \pi/2), \\ \bar{\varphi}_2(\theta) &\in C(\pi \leq \theta \leq 3\pi/2); \end{aligned} \quad (3)$$

$$\begin{aligned}\psi_1(y) &\in C(\tfrac{1}{2} \leq y \leq 1) \cap C^{(2,\alpha)}(\tfrac{1}{2} < y < 1), \\ \psi_2(y) &\in C(-1 \leq y \leq -\tfrac{1}{2}) \cap C^{(2,\alpha)}(-1 < y < -\tfrac{1}{2}), \end{aligned} \quad (4)$$

where $\psi'_1(y)$ as $y \rightarrow \frac{1}{2}$, $y \rightarrow 1$ and $\psi'_2(y)$ as $y \rightarrow -1$, $y \rightarrow -\frac{1}{2}$ may tend to ∞ of order lower than 1; $\varphi_1(\pi/2) = \psi_1(1)$, $\varphi_2(3\pi/2) = \psi_2(-1)$.

We shall show that the solution of boundary-value problem A exists, is unique, and can be written in explicit form.

$$u(x, y) = u_i(x, y), \quad (x, y) \in \Omega_i; \quad u(x, y) = u_i^*(x, y), \quad (x, y) \in \Omega_i^*; \quad i = 1, 2; \quad (5)$$

$$u_y(x, 0) = \begin{cases} \nu_1(x), & 0 < x < 1, \\ \nu_1^*(x), & -1 < x < 0; \end{cases} \quad u_x(0, y) = \begin{cases} \nu_2(y), & 0 < y < 1, \\ \nu_2^*(y), & -1 < y < 0; \end{cases} \quad (6)$$

$$u(x, x-1) = X_1(x), \quad \tfrac{1}{2} \leq x \leq 1;$$

$$u(x, x+1) = X_2(x), \quad -1 \leq x \leq -\tfrac{1}{2}, \quad (7)$$

where we shall assume that $X_1(x) \in C^{(2,\alpha)}(1/2 < x < 1)$, $X_2(x) \in C^{(2,\alpha)}(-1 < x < -1/2)$, while $X'_1(x)$ at $x = 1/2$, $x = 1$ and $X'_2(x)$ at $x = -1$, $x = -1/2$ may have singularities of order less than 1 (below it will be seen that $X_1(x)$ and $X_2(x)$ possess these properties).

The solution $u(x, y)$ of problem A in the domain Δ coincides with the solution of the following Gellerstedt boundary-value problem: find a function $u(x, y)$ having the properties: 1) $u(x, y) \in C(\bar{\Delta}) \cap C^{(1,0)}(\Delta)$; 2) $u_x(x, 0)$, $u_x(0, y)$, $u_y(x, 0)$, $u_y(0, y)$ may have singularities of order less than 1 at the points A, B, and O; 3) $u(x, y)$ is a regular solution of equation (1) in $\Delta \setminus (OA \cup OB)$, satisfying the boundary conditions $u|_{AB} = \varphi_1(\theta)$, $u|_{BC} = \psi_1(y)$, $u|_{DA} = X_1(x)$.

In paper ⁽²⁾ it was proved that there exists a unique solution of the formulated boundary-value problem, which can be written in explicit form, and that the functions (6) and (7) are connected by the relations:

$$\nu_i(x) = \frac{F_i(x)}{2} + \frac{1}{\pi} \int_0^1 N_1(x, t) F_i(t) dt - \frac{1}{\pi} \int_0^1 N_2(x, t) F_j(t) dt, \quad (8)$$

$$\nu_i^*(-x) = \frac{F_i^*(-x)}{2} + \frac{1}{\pi} \int_0^1 N_1(x, t) F_i^*(-t) dt - \frac{1}{\pi} \int_0^1 N_2(x, t) F_j^*(-t) dt, \quad (9)$$

where

$$N_k(x, t) = t \left(\frac{1-x^4}{1-t^4} \right)^{1/2} \left[\frac{1}{t^2 + \beta x^2} - \frac{t^2}{1 + \beta t^2 x^2} \right], \quad \beta = \begin{cases} -1, & k=1, \\ 1, & k=2; \end{cases} \quad (10)$$

$$F_1(x) = 2 \frac{d}{dx} \left[X_1 \left(\frac{x+1}{2} \right) - g_1(x) \right], \quad F_2(x) = 2 \frac{d}{dx} \left[\psi_1 \left(\frac{x+1}{2} \right) - g_2(x) \right]; \quad (11)$$

$$F_1^*(-x) = 2 \frac{d}{dx} \left[g_1^*(-x) - X_2 \left(-\frac{x+1}{2} \right) \right],$$

$$F_2^*(-x) = 2 \frac{d}{dx} \left[g_2^*(-x) - \psi_2 \left(-\frac{x+1}{2} \right) \right]; \quad (12)$$

$$g_i(x) = \frac{1}{2\pi} \int_0^{\pi/2} \varphi_1(\theta) \frac{\partial G_i}{\partial n} \Big|_{|\zeta|=1} d\theta, \quad g_i^*(-x) = \frac{1}{2\pi} \int_{\pi}^{3\pi/2} \varphi_2(\theta) \frac{\partial G_i}{\partial n} \Big|_{|\zeta|=1} d\theta; \quad (13)$$

$$G_1(x; \xi, \eta) = \ln |1 - \zeta^2 x^2| - \ln |\zeta^2 - x^2|, \quad G_2(x; \xi, \eta) = \ln |1 + \zeta^2 x^2| - \ln |\zeta^2 + x^2|,$$

$$\zeta = \xi + i\eta; \quad 0 < x < 1, \quad i, j = 1, 2, \quad i \neq j; \quad n \text{ is the inner normal.}$$

Relations (9) are obtained in the same way as (8)—by solving the Gellerstedt problem in the domain Δ^* with the data $\varphi_2(\theta)$, $\psi_2(y)$, and $X_2(x)$ on B^*A^* , A^*D , and B^*C , respectively.

By virtue of the continuity of the solution $u(x, y)$ of problem A in the closed domain $\bar{\Omega}$, we have

$$u_1(x, -x) = u_1^*(x, -x), \quad u_2(x, -x) = u_2^*(x, -x), \quad u_1(0, 0) = u_2^*(0, 0); \quad (14)$$

$$X_1(1/2) = \psi_2(-1/2), \quad X_2(-1/2) = \psi_1(1/2). \quad (15)$$

From d' Alembert's formula it is obvious that the functions u_1 and u_2 (u_1^* and u_2^*) can be written explicitly in terms of the functions $X_1, \nu_1, \psi_1, \nu_2$ (respectively $\psi_2, \nu_2^*, X_2, \nu_1^*$). Hence, on the basis of equalities (14) and (15), we conclude that

$$\begin{aligned} \nu_1(x) + \nu_2^*(-x) + \frac{d}{dx} \left[\psi_2 \left(-\frac{x+1}{2} \right) - X_1 \left(\frac{x+1}{2} \right) \right] &= 0, \\ \nu_2(x) + \nu_1^*(-x) + \frac{d}{dx} \left[X_2 \left(-\frac{x+1}{2} \right) - \psi_1 \left(\frac{x+1}{2} \right) \right] &= 0, \end{aligned} \quad (16)$$

$$2[\psi_2(-1/2) - \psi_1(1/2)] + X_2(-1) - X_1(1) + \int_0^1 [\nu_1(t) + \nu_1^*(-t)] dt = 0.$$

Eliminating from (8), (9), and (16) $v_i(x)$, $v_i^*(-x)$, $i = 1, 2$, and taking into account (11), (12), we obtain

$$\frac{2}{\pi} \int_0^1 N_1(x, t) \frac{d}{dt} \left[X_1 \left(\frac{t+1}{2} \right) \right] dt + \frac{2}{\pi} \int_0^1 N_2(x, t) \frac{d}{dt} X_2 \left(-\frac{t+1}{2} \right) dt = P_1(x), \quad (17)$$

$$\frac{2}{\pi} \int_0^1 N_1(x, t) \frac{d}{dt} X_2 \left(-\frac{t+1}{2} \right) dt + \frac{2}{\pi} \int_0^1 N_2(x, t) \frac{d}{dt} X_1 \left(\frac{t+1}{2} \right) dt = P_2(x), \quad (18)$$

$$\frac{2}{\pi} \int_0^1 ds \int_0^1 N_1(s, t) \frac{d}{dt} \left[X_1 \left(\frac{t+1}{2} \right) - X_2 \left(-\frac{t+1}{2} \right) \right] dt = P, \quad (19)$$

where

$$P_1(x) = \frac{d}{dx} [g_1(x) - g_2^*(-x)] + \frac{1}{\pi} \int_0^1 N_1(x, t) h_1(t) dt + \frac{1}{\pi} \int_0^1 N_2(x, t) h_2(t) dt, \quad (20)$$

$$P_2(x) = \frac{d}{dx} [g_1^*(-x) - g_2(x)] + \frac{1}{\pi} \int_0^1 N_1(x, t) h_2(t) dt + \frac{1}{\pi} \int_0^1 N_2(x, t) h_1(t) dt, \quad (21)$$

$$h_1(x) = 2 \frac{d}{dx} g_1(x) - F_2^*(-x), \quad h_2(x) = 2 \frac{d}{dx} g_1^*(-x) + F_2(x), \quad (22)$$

and P is a particular number which depends only on the prescribed functions ψ_1 , ψ_2 , φ_1 , and φ_2 .

Consequently, problem A has been reduced to finding functions $X_1\left(\frac{x+1}{2}\right)$, $X_2\left(-\frac{x+1}{2}\right) \in C^{(2,\alpha)}(0 < x < 1)$, satisfying the system of equalities (17), (18), (19) and the conditions (15), where the first derivatives of X_1 and X_2 may become infinite of order less than unity at $x = 0$ and $x = 1$.

It is clear that the functions X_1 and X_2 satisfying equalities (17) and (18) can be obtained from the solution of the equations

$$\frac{2}{\pi} \int_0^1 [N_1(x, t) \pm N_2(x, t)] \mu_i(t) dt = M_i(x), \quad i = 1, 2, \quad (23)$$

where

$$\mu_i(x) = \frac{d}{dx} \left[X_1\left(\frac{x+1}{2}\right) \pm X_2\left(-\frac{x+1}{2}\right) \right], \quad M_i(x) = P_1(x) \pm P_2(x), \quad (24)$$

with the plus sign taken for $i = 1$, and the minus sign for $i = 2$.

Substituting the values of $N_1(x, t)$, $N_2(x, t)$ from (10) into (23), after the change of variables

$$t = \tau^{1/4} (1 + \sqrt{1 - \tau^2})^{-1/4}, \quad x = y^{1/4} (1 + \sqrt{1 - y^2})^{-1/4}, \quad (25)$$

equations (23) take the form ²

$$\frac{1}{\pi} \int_0^1 \frac{\rho_i(\tau)}{\tau - y} d\tau = R_i(y), \quad i = 1, 2, \quad (26)$$

where

$$\rho_i(y) = \delta_i(x)\mu_i(x), \quad R_i(y) = \delta_i(x)M_i(x); \quad (27)$$

$$\delta_1(x) = (1+x^8)(1+x^4)^{-1}(1-x^4)^{-1/2}, \quad \delta_2(x) = (1+x^8)x^{-2}(1-x^4)^{-1/2}, \quad (28)$$

and x is related to y by equality (25).

It follows from (13) that $g_i(x)$ and $g_i^*(-x)$, for $0 < x < 1$, are analytic functions.

Taking (3) into account, we ascertain that $\frac{d}{dx}g_i(x)$, $\frac{d}{dx}g_i^*(-x)$ have finite limits as $x \rightarrow 0$, $x \rightarrow 1$. Hence, by virtue of the assumptions on the given functions $\psi_i(x)$ (see (4)), the equalities (11), (12), (20), (21), (22), (24), and the properties of integrals of Cauchy type [1], we obtain that $M_i(x) \in C^{(1,\alpha)}(0 < x < 1)$, $i = 1, 2$; moreover, at $x = 0$ and $x = 1$ these functions may tend to ∞ with order less than 1.

On the basis of equalities (25), (27), and (28) we now conclude that the functions $R_i(y) \in C^{(1,\alpha)}(0 < y < 1)$ and may tend to ∞ with order less than $1/4$ ($3/4$) in the case $i = 1$, and less than $3/4$ ($3/4$) in the case $i = 2$, when $y \rightarrow 0$ ($y \rightarrow 1$). From the same relations (24), (27), and (28) it follows that the solutions of equations (26), $\rho_i(y)$, on the interval $0 \leq y \leq 1$ must possess the same properties as $R_i(y)$. It is known that such solutions of equations (26) exist and can be written in quadratures [1].

Solving (26) and returning to the functions X_1, X_2 , by virtue of equality (19) and conditions (15), we ascertain that X_1 and X_2 are determined uniquely. Consequently, problem A is uniquely solvable.

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CITED LITERATURE

1. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.
2. M. M. Zainulabidov, *Differential Equations*, 5, No. 1 (1969).

Note: Figure translations are in progress. See original paper for figures.

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