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Abstract

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MATHEMATICAL PHYSICS

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ON THE SECOND ONE-DIMENSIONAL IMPERFECT THERMAL-CONTACT BOUNDARY-VALUE PROBLEM

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In the present paper we study the problem of heat propagation in a composite rod consisting of two different pieces, under imperfect thermal-contact conditions at the junction boundary of the component parts.

1°. Suppose that the composite rod occupies the interval $(0, l)$, the point $x = l/2$ is the junction point, the surface of the rod is thermally insulated, there are no sources or sinks in the body, and at the ends there is heat exchange with the surrounding medium.

Denote by $c_1, \rho_1, K_1, \kappa_1 = K_1/c_1\rho_1$ and $c_2, \rho_2, K_2, \kappa_2 = K_2/c_2\rho_2$ the heat capacity, density, thermal conductivity, and thermal diffusivity of the parts of the rod occupying, respectively, $(0, l/2)$ and $(l/2, l)$. Then the problem of heat propagation in such a rod is formulated as follows: it is required to find in the domain

$$S_T^{(i)} = \{(x, t), 0 < x < l/2; l/2 < x < l; 0 < t < T\} \quad (i = 1, 2) \quad (1)$$

a temperature distribution $u_i(x, t)$ ($i = 1, 2$) such that

$$\frac{\partial u_i}{\partial t} = \kappa_i \frac{\partial^2 u_i}{\partial x^2}, \quad (x, t) \in S_T^{(i)} \quad (i = 1, 2); \quad (2)$$

$$u_1(x, 0) = F_1(x), \quad 0 \leq x \leq l/2; \quad u_2(x, 0) = F_2(x), \quad l/2 \leq x \leq l; \quad (3)$$

$$\frac{\partial u_1(0, t)}{\partial x} = f_1(t); \quad \frac{\partial u_2(l, t)}{\partial x} = f_2(t), \quad 0 \leq t \leq T; \quad (4)$$

$$-K_1 \frac{\partial u_1(l/2, t)}{\partial x} = h[u_1(l/2, t) - u_2(l/2, t)], \quad 0 \leq t \leq T; \quad (5)$$

$$K_1 \frac{\partial u_1(l/2, t)}{\partial x} = K_2 \frac{\partial u_2(l/2, t)}{\partial x}, \quad 0 \leq t \leq T; \quad (6)$$

where the compatibility conditions are satisfied

$$F'_1(0) = f_1(0); \quad F'_2(l) = f_2(0); \quad K_1 F'_1(l/2) = h[F_2(l/2) - F_1(l/2)];$$

$$K_1 F'_1(l/2) = K_2 F'_2(l/2), \quad (7)$$

where $h = 1/R$ is the contact conductance, R is the contact resistance, $f_i(t) \in C^{(1)}$, $F_i(x) \in C^{(2)}$ ($i = 1, 2$).

We call problem (1)–(6) the second imperfect one-dimensional thermal-contact boundary-value problem, in contrast to the case where we have infinitely large values of h , for which a continuous transition of temperature through the thermal-contact boundary takes place. We call the latter the ideal thermal-contact boundary-value problem. The ideal problem was considered in ^(1–4) for parabolic equations and in ^(5–7) for elliptic equations.

We shall represent the solution of problem (1)–(6) in the form

$$u_i(x, t) = v_i(x, t) + w_i(x, t) \quad (i = 1, 2), \quad (8)$$

where $v_i(x, t)$ ($i = 1, 2$) are the solution of problem (1)–(6) with homogeneous boundary conditions (4), and $w_i(x, t)$ ($i = 1, 2$) is the solution of problem (1)–(6) with homogeneous initial conditions (3). For brevity, we shall speak of the latter problems as problems I and II. The uniqueness of the solution of problem I is easily proved.

2°. To find the unique solution of problem I, the Fourier method is applied, which leads to the function

$$v_{1,2}(x, t) = \sum_{n=0}^{\infty} D_n v_{1,2}^{(n)}(x, t) = \quad (9)$$

$$= \begin{cases} \sum_{n=0}^{\infty} D_n X_{1n}(x) T_n(t) = \sum_{n=0}^{\infty} D_n \left(a_{1n} \cos \frac{\lambda_n x}{\sqrt{\chi_1}} \right) \exp[-\lambda_n^2 t], & 0 \leq x \leq l/2, \\ \sum_{n=0}^{\infty} D_n X_{2n}(x) T_n(t) = \sum_{n=0}^{\infty} D_n \left(a_{2n} \cos \frac{\lambda_n x}{\sqrt{\chi_2}} + \sin \frac{\lambda_n x}{\sqrt{\chi_2}} \right) \exp[-\lambda_n^2 t], & l/2 \leq x \leq l, \quad 0 \leq t \leq T, \end{cases}$$

where a_{1n}, a_{2n} ($n = 0, 1, 2, \dots$) are known quantities, and λ_n ($n = 1, 2, \dots$) are the roots of the transcendental equation

$$\left(\frac{K_1 \lambda}{\sqrt{\chi_1}} \sin \frac{\lambda l}{2\sqrt{\chi_1}} - h \cos \frac{\lambda l}{2\sqrt{\chi_1}} \right) \sin \frac{\lambda l}{2\sqrt{\chi_2}} - \frac{K_1}{K_2} \sqrt{\frac{\chi_2}{\chi_1}} h \sin \frac{\lambda l}{2\sqrt{\chi_1}} \cos \frac{\lambda l}{2\sqrt{\chi_2}} = 0, \quad (10)$$

and the functions $X_{1n}(x), X_{2n}(x)$ ($n = 0, 1, 2, \dots$), when multiplied respectively by $\sqrt{c_1 \rho_1}$ and $\sqrt{c_2 \rho_2}$, become orthogonal in the sense that

$$c_1 \rho_1 \int_0^{l/2} X_{1n}(x) X_{1m}(x) dx + c_2 \rho_2 \int_{l/2}^l X_{2n}(x) X_{2m}(x) dx = 0 \quad \text{for } n \neq m. \quad (11)$$

Satisfying the initial conditions (3), by virtue of (11) we find:

$$D_k = -\frac{1}{\lambda_k^2} \left[K_1 \int_0^{l/2} F_1''(x) X_{1k}(x) dx + K_2 \int_{l/2}^l F_2''(x) X_{2k}(x) dx \right] \quad (k = 1, 2, \dots), \quad (12)$$

from which it is seen that the series in (9) for $t = 0$ converge absolutely and uniformly, and that the initial conditions (3) are indeed satisfied if the amplitudes D_k ($k = 1, 2, \dots$) are determined by formula (12).

3°. It is obvious that (9) satisfies the homogeneous boundary conditions (4) and the thermal-contact conditions (5)–(6). To prove that (9) satisfies (2) in the domain (1), taking into account the peculiarity of problem I ($\lambda_0 = 0$ is an eigenvalue with eigenfunctions $X_{10}(x) = 1, X_{20}(x) = 1$), we construct the generalized Green's function $G^*(x, s)$, and replace the Sturm-Liouville problem obtained by separation of variables for $\lambda \neq 0$, as in (5–7), by the equivalent integral equation

$$X(x) + \mu \int_0^l G^*(x, s) X(s) ds = 0, \quad \mu = \lambda^2,$$

and this makes it possible to prove the uniform convergence of the series obtained from (9), differentiated once with respect to t and twice with respect to x .

4°. We solve problem II by applying Duhamel's principle (8), which requires the preliminary solution of a certain thermal-contact boundary-value problem with constant boundary data, obtained from problem II by “freezing” the boundary data $f_1(t)$ and $f_2(t)$ for some value $t = \eta$. Problem II is a problem with stationary nonhomogeneity, and therefore it is expedient first to isolate the stationary solution and then to seek the deviation from this solution.

It turns out that in the present case this cannot be done directly, unlike the first and third thermal-contact boundary-value problems, since

In this case, by virtue of the boundary conditions (4), two constants of integration disappear which cannot be determined. To avoid this indeterminacy, denoting by $u_i^0(x, t)$ ($i = 1, 2$) the solution of the frozen problem, we introduce the function

$$q_i(x, t) = -K_i \partial u_i^0(x, t) / \partial x \quad (i = 1, 2), \quad (13)$$

and without particular difficulty transform the frozen problem into the following one, which we call the associated problem, which is the first ideal thermal-contact boundary-value problem, with stationary nonhomogeneity:

$$\begin{aligned} \partial q_1 / \partial t &= \chi_1 \partial^2 q_1 / \partial x^2, & 0 < x < l/2; \\ \partial q_2 / \partial t &= \chi_2 \partial^2 q_2 / \partial x^2, & l/2 < x < l; \\ q_1(x, 0) &= 0, \quad 0 \leq x \leq l/2; & q_2(x, 0) &= 0, \quad l/2 < x < l; \\ q_1(0, t) &= -K_1 f_1(\eta); & q_2(l, t) &= -K_2 f_2(\eta); & 0 \leq t \leq T; \\ q_1(l/2, t) &= q_2(l/2, t); & \partial q_1(l/2, t) / \partial x &= A q_2(l/2, t); & 0 \leq t \leq T, \end{aligned} \quad (14)$$

where A is a known constant.

5°. Setting $q_i(x, t) = \Phi_i(x) + \theta_i(x, t)$ ($i = 1, 2$), we split problem (14) into a stationary problem with nonhomogeneous boundary conditions and a nonstationary one with nonhomogeneous initial, but homogeneous boundary, conditions. Solving the stationary problem directly, and the nonstationary one, as in problem I, by applying the Fourier method, we find $\theta_i(x, t)$ and construct the solution of the "associated" problem $q_i(x, t)$ ($i = 1, 2$); substituting it into (13), we find $w_i^0(x, t)$ ($i = 1, 2$), which contain arbitrary functions $C_1(t)$ and $C_2(t)$.

6°. To determine these arbitrary functions, following (9), we introduce the means

$$\overline{w_1^0}(t) = \frac{2}{l} \int_0^{l/2} w_1^0(x, t) dx; \quad \overline{w_2^0}(t) = \frac{2}{l} \int_{l/2}^l w_2^0(x, t) dx; \quad 0 \leq t \leq T, \quad (15)$$

for which the relations are valid:

$$C_1 \rho_1 \frac{l}{2} \frac{d\overline{w_1^0}(t)}{dt} = f_1(\eta); \quad C_2 \rho_2 \frac{l}{2} \frac{d\overline{w_2^0}(t)}{dt} = f_2(\eta). \quad (16)$$

Integrating (16) and substituting the expressions $w_i^0(x, t)$ ($i = 1, 2$) into (15), we find four expressions, from which $C_1(t)$ and $C_2(t)$ are determined. Substituting the values found for $C_i(t)$ ($i = 1, 2$) into $w_i^0(x, t)$, after passing to dimensionless quantities we find the solution of the frozen problem in the form

$$w_{1,2}^0(\xi, \tau) = \begin{cases} w_1^0(\xi_1, \tau_1) = L_{11}(\xi_1, \tau_1)f_1(\eta) + L_{12}(\xi_1, \tau_1)f_2(\eta); & 0 \leq \xi_1 \leq 1; \\ w_2^0(\xi_2, \tau_2) = L_{21}(\xi_2, \tau_2)f_1(\eta) + L_{22}(\xi_2, \tau_2)f_2(\eta); & 1 \leq \xi_2 \leq 2, \end{cases}$$

where $L_{ij}(\xi_k, \tau_k)$ ($i, j, k = 1, 2$) have definite explicit expressions.

7°. Now from the last expression, by the Duhamel principle ⁽⁸⁾, the solution of problem II corresponding to the variable boundary data $f_1(t)$ and $f_2(t)$ can be written in the form

$$w_{1,2}(\xi, \tau) = \begin{cases} w_1(\xi_1, \tau_1) = \int_0^{\tau_1} \left\{ f_1(\gamma) \frac{\partial}{\partial \tau_1} L_{11}(\xi_1, \tau_1 - \gamma) + f_2(\gamma) \frac{\partial}{\partial \tau_1} L_{12}(\xi_1, \tau_1 - \gamma) \right\} d\gamma, & 0 \leq \xi_1 \leq 1; \\ w_2(\xi_2, \tau_2) = \int_0^{\tau_2} \left\{ f_1(\gamma) \frac{\partial}{\partial \tau_2} L_{21}(\xi_2, \tau_2 - \gamma) + f_2(\gamma) \frac{\partial}{\partial \tau_2} L_{22}(\xi_2, \tau_2 - \gamma) \right\} d\gamma, & 1 \leq \xi_2 \leq 2. \end{cases} \quad (17)$$

Reducing (9) to the same dimensionless quantities as (17), and adding, by the principle of superposition, to the latter, we obtain the solution of problem (1)–(7).

Remark. Let us note that problem (1)–(7) has also been investigated for the domain

$$S_T^{(i)} = \{(x, t); \chi_i(t) < x < \chi_{i+1}(t); 0 < t < T\} \quad (i = 1, 2), \quad (18)$$

where $\chi_i(t)$ ($i = 1, 2, \dots$) have no common points and satisfy a Hölder condition with exponent α ($1/2 < \alpha \leq 1$), and also in the case when the lateral surfaces of the rod are not thermally insulated and there is a source or sink of strength $Q_i(x, t)$ ($i = 1, 2$) inside the body. It is shown that the number of constituent parts can be increased.

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Note: Figure translations are in progress. See original paper for figures.

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