

ON RIEMANNIAN SPACES ADMITTING A GENERALLY RECURRENT SYMMETRIC TENSOR OF THE SECOND ORDER

V. S. Sobchuk

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.74541>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513

MATHEMATICS

V. S. SOBCHUK

ON RIEMANNIAN SPACES ADMITTING A GENERALLY RECURRENT SYMMETRIC TENSOR OF THE SECOND ORDER

(Presented by Academician A. D. Aleksandrov on 11 IX 1968)

1°. Definition. A symmetric tensor b_{ij} of the second order will be called **generally recurrent** if it satisfies the condition

$$b_{ij,l} = \bar{\lambda}_l g_{ij} + \lambda_i g_{jl} + \lambda_j g_{il} + \bar{\mu}_l b_{ij} + \mu_i b_{jl} + \mu_j b_{il}, \quad (1)$$

where g_{ij} is the metric tensor of the Riemannian space V_n ; $\bar{\lambda}_i, \lambda_i, \bar{\mu}_i, \mu_i$ are certain covariant tensors, which we shall call the **tensors of generalized recurrence**; the comma denotes covariant differentiation.

In particular, if $\bar{\lambda}_i = 0, \lambda_i = 0, \mu_i = 0$, then we obtain a recurrent tensor b_{ij} ⁽¹⁾. Condition (1) also includes the tensor characteristic of hypersurfaces of the second order ⁽²⁾ (for $\bar{\lambda}_i = 0, \lambda_i = 0, \bar{\mu}_i = \mu_i$), and the condition that the metrics g_{ij} and b_{ij} have common geodesics ⁽³⁾ (for $\lambda_i = 0, \bar{\lambda}_i = 0, \bar{\mu}_i = 2\mu_i$), and, finally, the characteristic condition that the vector field ξ_i determines a one-parameter group of projective transformations ⁽³⁾ (for $\bar{\mu}_i = 0, \mu_i = 0, \bar{\lambda}_i = 2\lambda_i, b_{ij} = \xi_{(i,j)}$).

In the present article we find all Riemannian spaces admitting a generally recurrent symmetric tensor of the second order, i.e., we find all metrics for which the system of equations (1) is compatible.

2°. Preliminary information. Among the eigenvalues $k_1, k_2, \dots, k_n$ of the tensor b_{ij} , i.e., among the roots of the equation $|kg_{ij} - b_{ij}| = 0$, there may also be multiple ones. Let a frame of eigenvectors $\eta_{a_1}^i$ be orthonormalized: $g_{ij}\eta_{a_1}^i\eta_{b_1}^j = \delta_{ab}$; then we also have $b_{ij}\eta_{a_1}^i\eta_{b_1}^j = k_a\delta_{ab}$, so that from (1) we obtain

$$b_{ij,l}\eta_{a_1}^i\eta_{b_1}^j\eta_{c_1}^l = 0, \quad a, b, c \neq . \quad (2)$$

When condition (2) is fulfilled, as is known ⁽³⁾, one can pass to new variables in such a way that we shall have

$$g_{ij}du^i du^j = \sum_{\alpha=1}^p \Phi_{\alpha}, \quad b_{ij}du^i du^j = \sum_{\alpha=1}^p k_{\alpha} \Phi_{\alpha}, \quad (3)$$

where $k_1, \dots, k_p$ are distinct roots, and the form Φ_{α} contains only the differentials of the variables $u^{i_{\alpha}}$ corresponding to k_{α} . The variables $u^1, u^2, \dots, u^n$ are divided into p groups $u^{i_1}, \dots, u^{i_p}$ (according to the number of distinct roots), and the number of variables in each group is equal to the multiplicity of the corresponding root, for example $u^{i_1} \equiv (u^1, \dots, u^{m_1})$, where m_1 is the multiplicity of the root k_1 . If among the roots $k_1, \dots, k_n$ there are simple ones, then we write them in the first places. Thus, if k_{α} is a simple root, then the group of variables $u^{i_{\alpha}}$ consists of the single variable u^{α} .

Equalities (3) mean that

$$g_{i_{\alpha}j_{\beta}} = b_{i_{\alpha}j_{\beta}} = 0, \quad \alpha \neq \beta; \quad b_{i_{\alpha}j_{\alpha}} = k_{\alpha}g_{i_{\alpha}j_{\alpha}}, \quad (4)$$

where $g_{i_{\alpha}j_{\alpha}}, b_{i_{\alpha}j_{\alpha}}$ depend, generally speaking, on all variables $u^1, \dots, u^n$.

3°. **Basic equations.** Computing directly the derivative $b_{i_{\alpha}j_{\alpha}, l_{\beta}}$, and also from (1) (using in both cases (4)), we find

$$(k_{\alpha} - k_{\beta})\partial \ln |g_{j_{\beta}, l_{\beta}}| / \partial u^{i_{\alpha}} = 2\lambda_{i_{\alpha}} + 2k_{\beta}\mu_{i_{\alpha}}. \quad (5)$$

Similarly, computing $b_{j_{\beta}l_{\beta}, i_{\alpha}}$ and $b_{i_{\alpha}i_{\alpha}, i_{\alpha}}$, we obtain

$$\partial k_{\beta} / \partial u^{i_{\alpha}} = \bar{\lambda}_{i_{\alpha}} + k_{\beta}\bar{\mu}_{i_{\alpha}}, \quad (6)$$

$$\partial k_{\alpha} / \partial u^{i_{\alpha}} = \bar{\lambda}_{i_{\alpha}} + k_{\alpha}\bar{\mu}_{i_{\alpha}} + 2\lambda_{i_{\alpha}} + 2k_{\alpha}\mu_{i_{\alpha}}.$$

Computing $b_{i_{\alpha}i_{\alpha}, j_{\alpha}}$, using (1), (4), and (7), we have

$$g_{i_{\alpha}i_{\alpha}}(\lambda_{j_{\alpha}} + k_{\alpha}\mu_{j_{\alpha}}) - g_{i_{\alpha}j_{\alpha}}(\lambda_{i_{\alpha}} + k_{\alpha}\mu_{i_{\alpha}}) = 0. \quad (7)$$

Interchanging the indices i_{α} and j_{α} and using the positive definiteness of the form $g_{ij}du^i du^j$, we find

$$\lambda_{i_{\alpha}} + k_{\alpha}\mu_{i_{\alpha}} = 0, \quad \text{if } k_{\alpha} \text{ is multiple.} \quad (8)$$

Consequently,

$$\partial k_{\alpha} / \partial u^{i_{\alpha}} = \bar{\lambda}_{i_{\alpha}} + k_{\alpha}\bar{\mu}_{i_{\alpha}}, \quad \text{if } k_{\alpha} \text{ is multiple.} \quad (9)$$

From (6), replacing β by γ and subtracting from (6), we obtain

$$\partial \ln |k_\beta - k_\gamma| / \partial u^{i_\alpha} = \bar{\mu}_{i_\alpha}. \quad (10)$$

From (10), replacing γ by δ and subtracting from (10), we obtain

$$\frac{\partial}{\partial u^{i_\alpha}} \ln |(k_\beta - k_\gamma)/(k_\beta - k_\delta)| = 0, \quad \alpha, \beta, \gamma, \delta \neq . \quad (11)$$

Here and below $p \geq 4$. Taking the logarithm and differentiating with respect to u^{l_β} of the identity

$$\frac{k_\alpha - k_\beta}{k_\alpha - k_\gamma} = \frac{k_\alpha - k_\beta}{k_\alpha - k_\delta} \frac{k_\alpha - k_\delta}{k_\alpha - k_\gamma}$$

and taking (11) into account, we find

$$\frac{\partial}{\partial u^{l_\beta}} \ln \left| \frac{k_\alpha - k_\beta}{k_\alpha - k_\gamma} \right| = \frac{\partial}{\partial u^{l_\beta}} \ln \left| \frac{k_\alpha - k_\beta}{k_\alpha - k_\delta} \right| \equiv \varphi_{\alpha l_\beta}(u^{i_\alpha}, u^{l_\beta}). \quad (12)$$

4°. **The functions** $\varphi_{\beta i_\alpha}$. Eliminating $\bar{\lambda}_{i_\alpha}$ from (6) and (9), we obtain

$$\partial \ln |k_\beta - k_\alpha| / \partial u^{i_\alpha} = \bar{\mu}_{i_\alpha}, \quad k_\alpha \text{ multiple}. \quad (13)$$

From (10) and (13) we find

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{k_\beta - k_\alpha}{k_\beta - k_\gamma} \right| = 0, \quad \text{if } k_\alpha \text{ is multiple,}$$

i.e., we have $\varphi_{\beta i_\alpha} = 0$, if k_α is multiple. Let $\varphi_{\beta \alpha} \neq 0$. Computing the ratio $\varphi_{\gamma \alpha} : \varphi_{\beta \alpha}$ by means of (12) and replacing the derivatives of $k_\alpha, k_\beta, k_\gamma$ by formulas (6) and (7), we obtain

$$\varphi_{\gamma \alpha} / \varphi_{\beta \alpha} = (k_\alpha - k_\beta) / (k_\alpha - k_\gamma). \quad (14)$$

Substituting (14) into (12), we find

$$\frac{\partial \varphi_{\beta \alpha}}{\partial u^{l_\beta}} = -\varphi_{\beta \alpha} \varphi_{\alpha l_\beta}. \quad (15)$$

The solution of the system (15) is the functions

$$\varphi_{\beta \alpha} = \varphi'_\alpha / (\varphi_\alpha - \varphi_\beta), \quad (16)$$

where φ_α is an arbitrary function of u_α , and $\varphi_\alpha = \text{const}$ if k_α is multiple. Thus the functions $\varphi_{\beta i_\alpha}$ are found from (16) both in the case when k_α, k_β are multiple, and when they are simple.

5°. **The curvatures k_α .** From (14) and (16) we obtain

$$(k_\alpha - k_\beta)/(\varphi_\alpha - \varphi_\beta) = (k_\alpha - k_\gamma)/(\varphi_\alpha - \varphi_\gamma), \quad (17)$$

i.e. the ratio $(k_\alpha - k_\beta)/(\varphi_\alpha - \varphi_\beta)$ does not depend on β . But

$$(k_\alpha - k_\beta)/(\varphi_\alpha - \varphi_\beta) = (k_\beta - k_\alpha)/(\varphi_\beta - \varphi_\alpha),$$

i.e., this ratio does not depend on α , and we shall denote it by F :

$$(k_\alpha - k_\beta)/(\varphi_\alpha - \varphi_\beta) = F. \quad (18)$$

From (18) we find $k_\alpha - \varphi_\alpha F = k_\beta - \varphi_\beta F \equiv \Phi$, or

$$k_\alpha = \varphi_\alpha F + \Phi, \quad (19)$$

where F and Φ are arbitrary functions.

6°. **Metric.** Adding (5), (6) and subtracting (7), we obtain

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{g_{j_\beta l_\beta}}{g_{j_\gamma l_\gamma}} \right| = -\bar{\mu}_{i_\alpha} - 2\mu_{i_\alpha}. \quad (20)$$

From (20), replacing β by γ and subtracting from (20), we obtain

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{g_{j_\beta l_\beta}}{g_{j_\gamma l_\gamma}} \right| = \frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{k_\alpha - k_\beta}{k_\alpha - k_\gamma} \right|. \quad (21)$$

From (17) and (21) we find

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{g_{j_\beta l_\beta}}{g_{j_\gamma l_\gamma}} \right| = \frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{\varphi_\alpha - \varphi_\beta}{\varphi_\alpha - \varphi_\gamma} \right|, \quad \alpha \neq \beta, \gamma. \quad (22)$$

Integrating (22), we have

$$\frac{g_{j_\beta l_\beta}}{g_{j_\gamma l_\gamma}} = B_{j_\beta l_\beta j_\gamma l_\gamma} \prod_{\sigma \neq \beta, \gamma} \left| \frac{\varphi_\sigma - \varphi_\beta}{\varphi_\sigma - \varphi_\gamma} \right|, \quad (23)$$

where $B_{j_\beta l_\beta j_\gamma l_\gamma}$ are functions only of $u^{k_\beta}, y^{m_\gamma}$. From (23) we obtain

$$B_{j_\beta l_\beta j_\gamma l_\gamma} = B_{j_\beta l_\beta j_\delta l_\delta} B_{j_\delta l_\delta j_\gamma l_\gamma}.$$

Consequently,

$$B_{j_\beta l_\beta j_\gamma l_\gamma} = B_{j_\beta l_\beta} / B_{j_\gamma l_\gamma}, \quad (24)$$

where $B_{j_\beta l_\beta}$ are functions only of u^{m_β} . From (23) and (24) we find

$$g_{j_\beta l_\beta} / B_{j_\beta l_\beta} \prod_{\sigma \neq \beta} |\varphi_\sigma - \varphi_\beta| = g_{j_\gamma l_\gamma} / B_{j_\gamma l_\gamma} \prod_{\sigma \neq \gamma} |\varphi_\sigma - \varphi_\gamma| = A.$$

Thus, we obtain

$$g_{j_\beta l_\beta} = A B_{j_\beta l_\beta} \prod_{\sigma \neq \beta} |\varphi_\sigma - \varphi_\beta|. \quad (25)$$

From (3) and (25) we find

$$g_{ij} du^i du^j = A \sum_{\alpha=1}^p \prod_{\sigma \neq \alpha} |\varphi_\sigma - \varphi_\alpha| ds_\alpha^2, \quad (26)$$

$$b_{ij} du^i du^j = A \sum_{\alpha=1}^p (\varphi_\alpha F + \Phi) \prod_{\sigma \neq \alpha} |\varphi_\sigma - \varphi_\alpha| ds_\alpha^2, \quad (27)$$

where $A > 0$, F , Φ are arbitrary functions; φ_α are functions only of u^α , and $\varphi_\alpha = \text{const}$ if k_α is multiple, $\varphi_\alpha \neq \varphi_\beta$, $\alpha \neq \beta$.

The metric

$$ds^2 = \sum_{\alpha=1}^p \prod_{\sigma \neq \alpha} |\varphi_\sigma - \varphi_\alpha| ds_\alpha^2$$

is called the Levi-Civita metric ⁽³⁾. Consequently, the metric (26) is conformal to the Levi-Civita metric. If, in a space with metric (26), the tensor b_{ij} is chosen according to (27), then it will evidently be generalized recurrent. Thus the following theorem has been proved.

Theorem. *A Riemannian space V_n , $n \geq 4$, admits a generalized recurrent symmetric tensor of the second order having p ($p \geq 4$) distinct proper values if and only if V_n is conformal to a Levi-Civita space.*

7. Some special cases.

From (10) and (19) we find $\bar{\mu}_i = \partial \ln |F| / \partial u^i$; hence, and from (6), we obtain $\bar{\lambda}_i = \partial \Phi / \partial u^i - \Phi \partial \ln |F| / \partial u^i$. From (5) and (7), taking into account the expressions found for $\bar{\lambda}_i$ and $\bar{\mu}_i$, we have

$$2\mu_i = -\partial \ln A / \partial u^i, \quad 2\lambda_{i_\alpha} = \varphi'_\alpha F + (F\varphi_\alpha + \Phi) \partial \ln A / \partial u^{i_\alpha}.$$

- 1) A semirecurrent tensor, i.e. $\lambda_i = 0$, $\mu_i = 0$. We find $A = \text{const}$, $\varphi_\alpha = \text{const}$. Consequently, the metric V_n is reducible. In particular, for a recurrent tensor we also have $\bar{\lambda}_i = 0$, since $\Phi = cF$, $c = \text{const}$. Consequently, $k_\alpha = F(\varphi_\alpha + c)$ —Datta's theorem⁽¹⁾ on the proportionality to one another of the proper values of a recurrent tensor.
- 2) If g_{ij}, b_{ij} are the first and second fundamental tensors of a hypersurface, then for $\lambda_i = 0$, $\bar{\lambda}_i = 0$, $\bar{\mu}_i = \mu_i$ we obtain the tensor characteristic of a hypersurface⁽²⁾. We find $\Phi = cF$, $c = \text{const}$, $A = \prod_{\sigma=1}^p f_\sigma^{-1}$, $f_\alpha = \varphi_\sigma + c$,

$$F = A^{-1/2} = \prod_{\sigma=1}^p f_\sigma^{1/2}.$$

Consequently,

$$k_\alpha = f_\alpha \prod_{\sigma=1}^p f_\sigma^{1/2}, \quad g_{i_\alpha j_\alpha} = B_{i_\alpha j_\alpha}(u^{m_\alpha}) \prod_{\sigma \neq \alpha} \left| 1 - \frac{f_\alpha}{f_\sigma} \right|,$$

i.e. we obtain the principal curvatures and the metric of a hyperquadric. At the same time we find

$$\mu_{i_\alpha} = \frac{1}{p+2} \frac{\partial \ln |K|}{\partial u^{i_\alpha}}, \quad K = \prod_{\sigma=1}^p k_\sigma.$$

- 3) For $\mu_i = 0$, $\bar{\mu}_i = 0$, $\bar{\lambda}_i = 2\lambda_i$, we obtain $F = \text{const}$, $A = \text{const}$,

$$\Phi = F \sum_{\sigma=1}^p \varphi_\sigma.$$

Consequently,

$$k_\alpha = f_\alpha + \sum_{\sigma=1}^p f_\sigma, \quad f_\alpha = F\varphi_\alpha, \quad ds^2 = \sum_{\alpha=1}^p \prod_{\sigma \neq \alpha} |f_\sigma - f_\alpha| ds_\alpha^2.$$

$$b_{ij} du^i du^j = \sum_{\alpha=1}^p \left(f_{\alpha} + \sum_{\sigma=1}^p f_{\sigma} \right) \prod_{\sigma \neq \alpha} |f_{\sigma} - f_{\alpha}| ds_{\alpha}^2, \quad \lambda_{i_{\alpha}} = \frac{\partial}{\partial u^{i_{\alpha}}} \left(\frac{1}{2} \sum_{\sigma=1}^p f_{\sigma} \right).$$

Thus, we obtain the Levi-Civita metric.

- 4) For $\bar{\lambda}_i = \lambda_i$, $\bar{\mu}_i = \mu_i$ we obtain $A = F^{-2}$, $\partial\Phi/\partial u^{i_{\alpha}} = \frac{1}{2}\varphi'_{\alpha}F - \varphi_{\alpha}\partial F/\partial u^{i_{\alpha}}$,
i.e. a generalization of our results ⁽⁴⁾ to the case of multiple curvatures.

Chernivtsi State University

Received

2 IX 1968

References

1. D. K. Datta, *Tensor*, **15**, No. 1, 61 (1964).
2. L. L. Verbickii, *Tr. seminara po vektorn. i tenzorn. analizu*, vol. 7, 1949.
3. A. S. Solodovnikov, *UMN*, **11**, no. 4 (70), 45 (1956).
4. V. S. Sobchuk, *Tr. seminara po vektorn. i tenzorn. analizu*, vol. 15, 1968.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.