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Abstract

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MATHEMATICS

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ON THE DECOMPOSABILITY OF SPACES

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The main results of this note are the proof of maximal decomposability of spaces of point-countable type and Theorems 5 and 12. Let X be a space.* The cardinal number $\Delta X = \min\{|U| : U \neq \Lambda \text{ and is open in } X\}$ is called the dispersion character of the space X . A dense-in-itself space X is called maximally decomposable if X is the sum of ΔX disjoint sets, each of which meets every nonempty open subset of X in at least ΔX points. Ščepin proved (see ⁽⁹⁾, Theorem 3) that if $\omega X \leq \Delta X$, then X is maximally decomposable. However, notions such as the π -base and the π -weight of a space (in the sense of V. I. Ponomarev**) turn out to be naturally connected with the notion of (maximal) decomposability. For example, if the topology of a space X contains a maximally decomposable topology that forms a π -base for X , then X is maximally decomposable.

Theorem 1. *If $\pi X \leq \Delta X$, then X is maximally decomposable.*

Assertion 1. *The following properties of an infinite space X are equivalent:*

- 1) X has a π -base $\mathcal{H}$ such that $x \in H \in \mathcal{H}$ implies $\pi(x, H) \leq |H|$;
- 2) X has a π -base such that, for each element H of it, $\pi H \leq |\overline{H}|$;
- 3) X has a π -base such that, for each element H of it, $\pi H \leq \Delta H$.

Definition 1. A space is called a π -space if it has one of properties 1)–3) of Assertion 1.

Theorem 2. *Every dense-in-itself π -space is maximally decomposable.*

Remark. π -spaces form the broadest class of spaces with respect to which it has been proved that every space of this class is maximally decomposable. The same is also true in the case of ordinary decomposability.***

Definition 2. A set $A \subseteq X$ is called characteristic in X if $\psi(x, X) = \chi(x, X)$, $\forall x \in A$.

Assertion 2. *Every characteristic space is a π -space and, consequently, every dense-in-itself characteristic space is maximally decomposable.*

Remark. The characteristic property of a bicomactum (and also of every locally bicomact space) was proved in the memoir of Aleksandrov–Uryson ((¹), pp. 36 and 65).

Theorem 3. *If a regular space X is the sum of an arbitrary number of its characteristic subspaces, each of which has countable character in X , then X is characteristic.*

* All spaces discussed below are assumed to be Hausdorff.

** A π -base of a space is a system of open sets dense in it; the π -weight πX of a space X and the π -character $\pi(x, X)$ of a point $x \in X$ in the space X are defined naturally (see (⁸), Definition 6.1).

*** For the definition of MI , SI , indecomposable and decomposable spaces, see ((¹¹)); for the definition of an R -minimal space and of a space minimal with respect to a set dense in it, see ((⁶)).

Corollary 1. *Every regular space of point-countable type is characteristic. In particular, all completely regular p -spaces and spaces complete in the sense of Čech are characteristic *.*

Together with V. V. Filippov the following has been proved.

Theorem 4. *The sum of any number of G_δ -subsets of a characteristic space is characteristic.*

By virtue of one of A. V. Arhangel'skii's theorems (see ((²), Theorem 3.13), the characteristic property of spaces of point-countable type can also be obtained from this, though under the condition of their complete regularity. It is easy to see that every space each point of which has a base ordered by inclusion is also characteristic. Thus, all known classes of spaces for which (maximal) decomposability has been proved ** are contained in the class of all characteristic spaces.

A space Y , defined on the same set of points as the space X , is called a relaxation of the space X (expansion, in Hewitt's terminology) if the identity mapping of Y onto X is continuous. Let Q be an ideal *** of subsets of X . We define a relaxation of the space X by taking as neighborhoods of each of its points x all sets of the form $(U \setminus q) \cup x$, where U is a neighborhood of the point x in X and $q \in Q$. This relaxation will be called the Q -relaxation of the space X ****. We shall need the following ideals: $I_\tau = \{A \subseteq X : |A| < \tau\}$, where $\tau \geq \aleph_0$; S, N, N_σ are the systems of all scattered, nowhere dense sets, sets of first category of the space X ; $\mathfrak{A}$ is the set of all subsets of the set A .

Assertion 3. *If X is a proper Q -relaxation, dense in itself, of some space, then X is nonregular.*

Example 1 of a π -space that is not characteristic. This is the space H from Alexandrov–Urysohn's memoir ((¹), p. 9. Recall that H is the $I_{\aleph_1}$ -relaxation of the real line. H is dense in itself and nonregular. For every $x \in H$,

$$\psi(x, H) = \aleph_0 < \pi(x, H) = \chi(x, H) = c = 2^{\aleph_0}.$$

Let us note one more interesting property of this space: every discrete-in-itself subset of H is at most countable and, therefore, closed. Thus, H gives us an example of a space not possessing Arhangel'skii's property (see ⁽⁴⁾, question 4⁰). This property of the space H was also noted by B. A. Efimov.

Example 2 of a decomposable space which is maximally decomposable if $c = \aleph_1$, and is not a π -space if the following hypothesis is true: for every family of sets of first category on the real line R , having cardinality c , there exists a subset of R of first category not contained in any of the elements of the given family. Such a space is the I_N -relaxation of R .

Hewitt (see ⁽¹¹⁾, Definition 7) introduced the concept of a τ -maximal space. This is a space X such that $\Delta X \geq \tau \geq \aleph_0$, but for every proper relaxation Y of it, $\Delta Y < \tau$. For $\tau = \aleph_0$ we obtain—

* Spaces of point-countable type were defined by A. V. Arhangel'skii as spaces admitting a covering by bicomponents of countable character (see ⁽²⁾, Definition 3.8). A. V. Arhangel'skii also gave the definition of a p -space and proved that every space complete in the sense of Čech is a p -space, while every completely regular p -space is a space of point-countable type (see ⁽³⁾, Definition 1.4, Theorems 2.3 and 2.6). Concerning this corollary, see also ⁽¹²⁾, Proposition 3.

** The decomposability of locally bicomponent spaces and of spaces each point of which has a base ordered by inclusion was proved by Hewitt (see ⁽¹¹⁾, Theorems 47 and 48). The maximal decomposability of these spaces was proved by Sidorenko (see ⁽⁹⁾, Theorems 7 and 8).

*** A family Q of sets is called an ideal if $q' \subset q \in Q$ implies $q' \in Q$, and $q, q' \in Q$ implies $q \cup q' \in Q$.

**** The topology of the Q -relaxation coincides with the Q -topology studied by Vaidyanathaswamy in ⁽⁵⁾.

we obtain the definition of a minimal space (see also ⁽⁶⁾, definition 2)*. Sidor and Pearson (see ⁽¹⁰⁾, p. 42) called a τ -maximal space for $\tau = \Delta X$ τ -minimal. We shall not discuss these definitions, since the definitions of a τ -maximal space for $\tau > \aleph_0$, and still more of a τ -minimal space, are made superfluous by the following

Theorem 5. *Every τ -maximal space is minimal.*

Proof. Let X be τ -maximal and let τ_0 be the least among all infinite cardinal numbers τ for which X is τ -maximal. Denote by Y the I_{τ_0} -relaxation of the space X . Then $\Delta Y = \Delta X$, and hence Y is not a proper relaxation of X . But then every subset of X of cardinality less than τ_0 is discrete in X . It follows

easily from this that for every proper relaxation Z of the space X we have $\Delta Z < \aleph_0$, and therefore $\tau_0 = \aleph_0$, i.e. X is minimal.

Theorem 6. *Every dense-in-itself space X has a minimal relaxation Y for which $\Delta Y = \Delta X$. For every $\tau \geq \aleph_0$ there exists a minimal space Z for which $\Delta Z = |Z| = \tau$.*

Proof. Hewitt (see ⁽¹¹⁾, theorem 12) proved that every space X , $\Delta X \geq \tau \geq \aleph_0$, has a τ -maximal relaxation, but by Theorem 5 this means nothing other than that every space X , $\Delta X \geq \tau \geq \aleph_0$, has a minimal relaxation Y for which $\Delta Y \geq \tau$. In order not to be tied to τ -maximal spaces, we give another (and just as trivial) proof, based only on the existence of minimal relaxations^{**}: the required space is the minimal relaxation of the $I_{\Delta X}$ -relaxation of the space X .

For the proof of the second assertion of the theorem, note that, in proving the existence of τ -maximal spaces for every $\tau \geq \aleph_0$, Hewitt (see ⁽¹¹⁾, theorem 13) refers to examples constructed by Pospíšil of (hereditarily normal) spaces of arbitrary infinite dispersion character. It remains for us to do the same.

Put

$$pX = \min\{|P| : P \neq \Lambda \text{ and is dense in itself in } X\}.$$

Corollary 2. *Every dense-in-itself space X for which $pX < \Delta X$ has nonhomeomorphic minimal relaxations.*

Theorem 7. *Every decomposable space has a minimal relaxation containing an equipotent discrete set.*

Theorem 8. *If a dense-in-itself space is not an MI, then it has an irregular MI-relaxation.*

Proof. Let^{***} $A \cap A' \neq \Lambda$ and $X \setminus A = P$ be dense in X . Denote by $X(P)$ the relaxation of X whose topology consists of all sets of the form $G \cup (P \cap U)$, where G and U are open in X . A relaxation Y of the space $X(P)$ is defined as follows: if $x \in A$, then the neighborhoods of the point x in Y are the same as in $X(P)$; if $x \in P$, then the neighborhoods of the point x in Y are the same as in the minimal relaxation of the space P . The required space is the S -relaxation of the $\mathfrak{A}$ -relaxation of the space Y .

Assertion 4. *If X is an MI, then $|X| \leq wX$.*

Corollary 3. *If X is an MI and $2^{sX} \leq |X|$, then $wX = |X|$.*

Proof. Let U be dense in X and $|U| = sX$. If $x \in U$, then $\chi(x, X) = \chi(x, U) \leq 2^{|U|} \leq |X|$. If $x \in X \setminus U$, then $U \cup x$ is open and again $\chi(x, X) = \chi(x, U \cup x) \leq 2^{|U \cup x|} \leq |X|$. Hence $wX \leq |X|$.

* The existence of minimal spaces was proved by Hewitt (see ⁽¹¹⁾, theorem 12) and Katětov (see ⁽⁶⁾, theorem 1).

** The proof of this fact “in pure form” should be looked up in Katětov.

*** A' is the set of limit points of the set A .

Theorem 9. Every dense-in-itself space X has a minimal relaxation Y for which $sY = sX$.

Proof. Let $A \subseteq X$, $[X \setminus A] = X$, $|X \setminus A| = sX$. The desired space is the minimal relaxation of the $\mathfrak{A}$ -relaxation of the space X .

Corollary 4. Every dense-in-itself space X for which $2^{sX} \leq |X|$ has a minimal relaxation Y for which $sX = sY < wY = |Y|$.

Corollary 5. There exist nonregular minimal spaces.

Assertion 5. The following properties of the space X are equivalent:

- 1) if U is open and $U \subseteq A \subseteq [U]$, then A is open in X ;
- 2) every closed subset of X is the sum of a clopen set and a discrete subset of X ;
- 3) if $x \in A'$, then $x \in \text{int}[A]$;
- 4) if a point $x \in X$ is not isolated in X , then X is minimal with respect to $X \setminus x$.

Definition 3. A space X is called **globally disconnected** if it possesses one of properties 1)–4) of Assertion 5.

The following theorems, together with Padmavally's theorem (see (7), Theorem 1) stating that the N -relaxation SI is MI , clarify the relations between indecomposable spaces.

Theorem 10. An extremally disconnected MI space is minimal.

Corollary 6. A dense-in-itself space is minimal if and only if it is minimal with respect to every dense-in-itself subset.

Theorem 11. A globally disconnected SI space is minimal.

Theorem 12. The S -relaxation of an R -minimal space is minimal and is its unique minimal relaxation.

Corollary 7. An R -minimal MI space is minimal.

Unfortunately, the existence of a regular minimal space still remains an unsolved problem, equivalent, by virtue of Theorem 12, to the following: does there exist an R -minimal space in which every discrete-in-itself subset is closed? In fact, as far as we know, even the following problem has no solution at present: does there exist a regular dense-in-itself space in which every discrete-in-itself subset is closed? One can only note that, by virtue of Assertion 3, such a space cannot be obtained by an S -relaxation, and from a negative solution of this problem it would follow that all minimal spaces (and even all MI spaces) are nonregular.

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