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Abstract

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MATHEMATICS

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ON THE COMPLETENESS OF SYSTEMS OF ANALYTIC FUNCTIONS GENERATED BY SUCCESSIVE DERIVATIVES OF ANALYTIC FUNCTIONS

Let D be an arbitrary finite or infinite simply connected domain of the z -plane, and let $A[D]$ be the class of all functions analytic in the domain D . In addition, let the system of functions $\{\varphi_n(z)\}$, analytic in the domain D , be complete in the domain D , i.e., for every function $f(z)$ from the class $A[D]$ there exists a sequence of linear combinations

$$\sum_{n=1}^{N_k} C_{n,k} \varphi_n(z)$$

with constant coefficients, which converges uniformly to the function $f(z)$ in any closed domain $\bar{D}_1$ situated inside the domain D .

Suppose that the function $F(z, u)$ is defined in the form of the series

$$F(z, u) = \sum_{n=0}^{\infty} \varphi_n(z) u^n \quad (1)$$

under the condition that the series (1) converges uniformly for all z and u situated respectively in the given domains D and G , where G is also a simply connected domain containing the point $u = 0$.

In the present note we report results of an investigation of the completeness of the system of functions

$$[\partial^n F(z, u) / \partial u^n]_{u=\alpha_n} = \partial^n F(z, \alpha_n) / \partial u^n \quad (2)$$

under various assumptions concerning the nature of the generating function $F(z, u)$ and the sequence of complex numbers $\{\alpha_n\}$.

1. In what follows we shall use the following completeness criterion:

A system of regular functions $\{f_k(z)\}$ is complete in the domain D if every function $\varphi_m(z) \in A[D]$ of the complete system $\{\varphi_m(z)\}$ is approximated arbitrarily well in the domain D by linear combinations of the form

$$\sum_{n=1}^{N_k} C_{n,k} f_n(z),$$

i.e., the inequalities hold:

$$\left| \varphi_k(z) - \sum_{n=1}^{N_k} C_{n,k} f_n(z) \right| < \varepsilon \quad (z \in D), \quad (3)$$

where $\varepsilon > 0$ is an arbitrarily small number.

In the investigation of the completeness of the system of functions (2) we use the Abel-Goncharov interpolation formula* for the function $F(z, u)$, with fixed z ($z \in D$), with nodes

$$\alpha_0, \alpha_1, \dots, \alpha_n, \dots \quad (\alpha_n \in G, \quad n = 0, 1, 2, \dots),$$

which has the form

$$F(z, u) = \sum_{k=0}^n \gamma_k(u) \left[\frac{\partial^k F(z, u)}{\partial u^k} \right]_{u=\alpha_k} + R_n(z, u), \quad (4)$$

* For the necessary information on the Abel-Goncharov interpolation process, see ⁽¹⁾.

where

$$\gamma_k(u) = \int_{\alpha_0}^u dt_1 \int_{\alpha_1}^{t_1} dt_2 \cdots \int_{\alpha_{k-1}}^{t_{k-1}} dt_k, \quad (5)$$

$$R_n(z, u) = \int_{\alpha_0}^u dt_1 \int_{\alpha_1}^{t_1} dt_2 \cdots \int_{\alpha_n}^{t_n} \frac{\partial^{n+1} F(z, t_{n+1})}{\partial t_{n+1}^{n+1}} dt_{n+1}. \quad (6)$$

Differentiating equality (4) m times with respect to u , and then putting $u = 0$, we find

$$\varphi_m(z) = \sum_{k=m}^n B_{m,k} \frac{\partial^k F(z, \alpha_k)}{\partial u^k} + R_{n,m}(z), \quad (7)$$

where

$$R_{n,m}(z) = \left[\frac{\partial^m R_n(z, u)}{\partial u^m} \right]_{u=0}.$$

Hence, and from inequality (3), it is clear that the system of functions (2) is complete in that domain G in which all the series

$$\sum_{k=m}^{\infty} B_{m,k} \frac{\partial^k F(z, \alpha_k)}{\partial u^k} \quad (m = 0, 1, 2, \dots) \quad (8)$$

converge uniformly. Consequently, the conditions under which the series (8) converge uniformly are at the same time conditions for the completeness of the system of functions (2) in the domain G . By these considerations we establish certain assertions on the completeness of the system of functions (2) under various assumptions concerning the nature of the function $F(z, u)$ and the sequence of numbers $\{\alpha_n\}$. From these assertions there follows a number of results of other authors on the completeness of systems of functions of the form $\{z^n f^{(n)}(\alpha_n z)\}$ under various assumptions concerning $f(z)$ and the sequence of numbers $\{\alpha_n\}$. One of the main results of the present paper is:

Theorem 1. Let the sequence of complex numbers $\{\alpha_n\}$ be such that $|\alpha_n| \leq r$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, and the series $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|$ converges. Moreover, let

$$F(z, u) = \sum_{n=0}^{\infty} \varphi_n(z) u^n$$

be an analytic function of z in a finite domain D and of u in the finite disk $|u| \leq r$, where r is a finite number.

Then the system of analytic functions $\{\partial^m F(z, \alpha_m) / \partial u^m\}$ is complete in the domain D .

2. In particular, if

$$f(z) = \sum_{n=0}^{\infty} C_n z^n$$

is an analytic function in the disk $|z| < R$ and $C_n = 0$ ($n = 0, 1, 2, \dots$), then, choosing $\varphi_n(z) = C_n z^n$, we find

$$F(z, u) = \sum_{n=0}^{\infty} C_n z^n u^n = f(zu)$$

for $|u| \leq 1$. Moreover, we have

$$\frac{\partial^k F(z, \alpha_k)}{\partial u^k} = z^k f^{(k)}(\alpha_k z) \quad (k = 0, 1, 2, \dots).$$

Thus, from Theorem 1 there follows the assertion:

Corollary 1. Let $f(z)$ be an analytic function in the disk $|z| < R$, with Taylor coefficients distinct from zero, $f^{(n)}(0) \neq 0$ ($n = 0, 1, 2, \dots$), and let a sequence of complex numbers $\{a_n\}$ be such that

$$|a_n| \leq 1, \quad \lim_{n \rightarrow \infty} a_n = 0$$

and the series

$$\sum_{n=1}^{\infty} |a_n - a_{n-1}|$$

converges.

Then the system of functions $\{z^n f^{(n)}(a_n z)\}$ is complete inside the disk $|z| < R$.

This assertion was first proved by another method by A. I. Markushevich ⁽²⁾, from whose result, in the case $a_n = 1$ ($n = 0, 1, 2, \dots$), there follows the following assertion, proved by the author in ⁽³⁾.

Corollary 2. If $f(z)$ is an analytic function in the finite disk $|z| < R$, with Taylor coefficients distinct from zero, $f^{(n)}(0) \neq 0$ ($n = 0, 1, 2, \dots$), then the system of functions $\{z^n f^{(n)}(z)\}$ is complete in the same disk $|z| < R$.

3. We now consider the case where $F(z, u)$ is an analytic function of z in a finite or infinite simply connected domain D and an entire analytic function of u , i.e. the series (1) converges uniformly for all z in the domain D and for all u . In addition, the sequence of complex numbers $\{a_n\}$ has the property

$$0 < |a_0| < |a_1| < \dots < |a_m| < \dots < |a_n| < \dots$$

and

$$\lim_{n \rightarrow \infty} |a_n| = \infty.$$

Theorem 2. Let $F(z, u)$ be an analytic function of z in a finite or infinite simply connected domain D and an entire function of u , with maximum modulus

$$M(F; D; r) = \max_{z \in D, |u| \leq r} |F(z, u)|.$$

Moreover, let $n(r)$ be the number of points of the sequence $\{S_n\}$ lying in the interval $(0, r)$, where

$$S_n = |a_0| + \sum_{k=1}^n |a_k - a_{k-1}|.$$

Then the system of analytic functions $\{\partial^n F(z, a_n)/\partial u^n\}$ is complete in any closed domain $\overline{D}_1$ situated inside the domain D , if the inequality

$$\log M(F; D; r/\theta) < C(\theta)n(r),$$

is satisfied, where

$$C(\theta) < \log \frac{1-\theta}{\theta}, \quad 0 < \theta < \frac{1}{2}.$$

From Theorem 2 there follow the following assertions:

Corollary 1. Let the function $F(z, u)$ be defined by the series (1), converging uniformly for all $z \in D$ and all u , and let the functions $\varphi_n(z)$ satisfy the conditions:

$$\lim_{n \rightarrow \infty} n^{1/\rho} |\varphi_n(z)|^{1/n} = (\varepsilon \rho)^{1/\rho} |z| \quad (z \in D).$$

Further, let $n(r)$ be the number of points of the sequence $\{S_n\}$ lying on the segment $(0, r)$, where

$$S_n = |a_0| + \sum_{k=1}^n |a_k - a_{k-1}|.$$

Moreover, let the numbers S_n satisfy the condition

$$\lim_{n \rightarrow \infty} \frac{n(r)}{S_n^\mu} = \nu > 0.$$

Then the system of functions $\{\partial^n F(z, \alpha_n)/\partial u^n\}$ is complete in the whole domain D for $\rho < \mu$, and is complete in the domain G_ω , which is the intersection of the domain D with the disk

$$|z| < \frac{\omega}{\omega+1} \left(\frac{\gamma}{\sigma} \log \frac{1}{\omega} \right)^{1/\rho},$$

for $\rho = \mu$, where ω is the positive root of the equation

$$\omega^\rho e^{\omega+1} = 1.$$

Corollary 2. Let

$$f(z) = \sum_{n=0}^{\infty} C_n z^n$$

be an entire analytic function with nonzero Taylor coefficients: $C_n \neq 0$ ($n = 0, 1, 2, \dots$), and with maximum modulus

$$M(f; r) = \max_{|z| \leq r} |f(z)|.$$

Moreover, let $n(r)$ be the number of points of the sequence $\{S_n\}$ lying in the interval $(0, r)$, where

$$S_n = |\alpha_0| + \sum_{k=1}^n |\alpha_k - \alpha_{k-1}|.$$

Then the system of functions $\{z^n f^{(n)}(\alpha_n z)\}$ is complete in any finite disk $|z| \leq R$, if the inequality

$$\log M(f; rR/\theta) < C(\theta)n(r),$$

is satisfied, where

$$C(\theta) < \log \frac{1-\theta}{\theta}, \quad 0 < \theta < \frac{1}{2}.$$

The last assertion was proved in the author's paper (4), written jointly with Ch. G. Atamaliyeva, from which there follows a number of concrete assertions previously proved by other authors (see also (3), when $f(z) = e^z$).

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